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INVARIANT METHODS IN CLASSICAL DIFFERENTIAL GEOMETRY*

BY W. C. GRAUSTEIN

There are two well known developments of classical differential geometry by invariant methods which are systematic and comprehensive, namely those of Cesàro† and Ricci.‡ Ricci's tensor analysis is so familiar as to need no detailed description, and the methods of Cesàro may perhaps best be set forth by comparing them with those of Ricci. In making this comparison there are two questions of major importance to consider, the one having to do with the point of departure of the theory, and the other with the method of differentiation.

Ricci based his developments on an analytic representation of the manifold under consideration. On the other hand, it was a matter of principle with Cesàro to get along without a fixed coordinate system to which to refer the manifold. Perhaps it is this strict adhesion to purely intrinsic geometry that has stood in the way of a more general adoption of Cesàro's methods. In any case, the opinion seems to be generally held that a more powerful theory is obtainable by combining the purely intrinsic methods with invariant methods based on representations by means of parameters or coordinates.

In contrast to Ricci's method of covariant differentiation, Cesàro employs the older method of intrinsic differentiation, differentiation along a curve with respect to the arc of the curve.§ It is probably the general opinion that here, too, Ricci's choice is the better one. This opinion is undoubtedly justifiable in the case of manifolds of higher dimensionality, but it is a question whether it can be upheld in the case of the manifolds of classical differential geometry.

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* An address presented to the Society at the request of the program committee, April 18, 1930.

† *Lezioni di Geometria Intrinseca*, 1896.

‡ *Lezioni sulla Teoria delle Superficie*, 1898; also, Ricci and Levi-Civita, *Méthodes de calcul différentiel et absolu et leurs applications*, Mathematische Annalen, vol. 54 (1901), pp. 125-201.

§ The method was used earlier by various writers, notably, Lamé, Bonnet, Gilbert, and Enneper.

In comparing the two methods in this case, let us, for example, consider the theory of a surface in euclidean 3-space. Fundamental in this theory are the three quadratic differential forms of the surface, and linear differential forms which, when equated to zero, represent families of curves on the surface. From the point of view of invariant theory, the important problem consists in finding the invariants of this system with respect to changes of parameters. We pass over the algebraic invariants of the system,—they are readily found. It is invariants depending on further differentiation which interest us.

Intrinsic differentiation of invariants yields new invariants as a result. On the other hand, covariant differentiation does not give rise immediately to new invariants. Applied to invariants or to tensors, it yields first new tensors, and only secondarily, by application of algebraic processes to these tensors, does it give rise to new invariants. It is in this way, for example, that geodesic curvature is introduced.

Equations expressed in terms of the components of tensors and their covariant derivatives are not, in general, invariant. This is the case, for example, for the equations corresponding to those of Codazzi. As a matter of fact, Ricci does not leave final results, such as these, in tensor form, but reverts to the use of actual invariants and their intrinsic derivatives.

But the intrinsic derivatives are, in themselves, sufficient. Covariant differentiation is only an intermediate process which is not only unnecessary, but also disadvantageous in that, in dealing essentially with tensors rather than with invariants, it requires the use of non-invariant quantities and equations.

We have noted that Cesàro's use of intrinsic differentiation was divorced from any fixed analytic representation of the surface. The introduction of intrinsic differentiation in connection with a parametric representation of the surface has been carried out by various authors, notably, Lilienthal,* Knoblauch,† and Blaschke.‡ However, in none of these treatments

* *Über geodätische Krümmung*, Mathematische Annalen, vol. 42 (1893), pp. 511–516; *Vorlesungen über Differentialgeometrie*, vol. II, 1913, Part IV.

† *Grundlagen der Differentialgeometrie*, 1913, pp. 56, 89, 191, 201, 205, 255, 351, 389, 522, 572.

‡ *Vorlesungen über Differentialgeometrie*, vol. I, 3d edition, 1930, pp. 123–145, 181–190, 289–292.

is full use made of the connection with the parametric representation. In fact, the importance of this connection for the development of a comprehensive and powerful theory seems not to have been completely realized.

The theory set forth in this paper aims to emphasize this connection, as well as to show the advantages of the use of directional derivatives. It also introduces two further tools, Jacobians and a modified form of intrinsic differentiation. Jacobians are employed in establishing the connection with the parametric representation, in obtaining invariant vectors and functions, and in introducing directional derivatives. The modified directional derivatives cooperate with the ordinary directional derivatives to yield a simpler, and, it is hoped, a more powerful, theory.*

The application of the theory has been restricted here to surfaces in euclidean 3-space. It will be evident, however, that the methods employed have a wider field of usefulness.

A. The Use of Jacobians

1. *Invariant Vectors.* Let $x=x(u, v)$:

$$x_1 = x_1(u, v), \quad x_2 = x_2(u, v), \quad x_3 = x_3(u, v)\dagger$$

be a real surface, S, referred to a system of real curves, and let

$$u = u(u', v'), \quad v = v(u', v')$$

be an arbitrary change of parameters, with Jacobian $\Delta \neq 0$.

Let $\phi(u, v)$ and $\psi(u, v)$ be two functions of u and v, and denote their Jacobian by $[\phi, \psi]$:

$$[\phi, \psi] = \phi_u\psi_v - \phi_v\psi_u.$$

If ϕ and ψ are transformed by (1) into ϕ' and ψ' so that $\phi' \equiv \phi$ and $\psi' \equiv \psi$, then

$$[\phi', \psi'] = \Delta[\phi, \psi].$$

* The theory, as set forth, is a revision of a recent paper by the author, *Méthodes invariantes dans la géométrie infinitésimale des surfaces*, Mémoires de l'Académie Royale de Belgique (Classe des Sciences), (2), vol. 11 (1929). The reader is referred to this paper for references concerning the details, and also for examples illustrating the applications of the theory.

† It is assumed that all functions are real, single-valued, and analytic throughout a certain domain of the real variables u, v.

Thus, the Jacobian of two absolutely invariant functions is a relative invariant of weight unity.

A vector whose components are all relative invariants of the same weight is said to be a relatively invariant vector.

If the components of a vector are the Jacobians of functions which are absolutely invariant with respect to an arbitrary change of parameters, the vector is relatively invariant.

A great majority of the important relatively invariant vectors have components formed essentially in this fashion.

2. *Invariant Vectors Normal to the Surface.* The familiar vector*

$$\alpha = \widehat{x_u x_v}$$

is a relatively invariant vector of weight one:

$$\alpha' = \Delta\alpha.$$

It gives rise to the unit vector, ζ, normal to the surface:

$$\zeta = \frac{\alpha}{(\alpha \mid \alpha)^{1/2}}.$$

Since

$$(\alpha' \mid \alpha')^{1/2} = \pm\, \Delta(\alpha \mid \alpha)^{1/2},$$

$$\zeta' = \pm\, \zeta,$$

where the plus sign or the minus sign is to be taken according as $\Delta > 0$ or $\Delta < 0$. Hence ζ is an absolutely invariant vector, except perhaps for a change of signs.

From x and ζ may be formed the two further vectors,

$$\gamma = \widehat{\zeta_u \zeta_v}, \qquad \beta = -\widehat{x_u \zeta_v} - \widehat{\zeta_u x_v},$$

normal to the surface.

The components of the three vectors α, β, γ may all be expressed in terms of Jacobians of invariant functions. We have:

$$(1) \quad \begin{cases} \alpha_1 = [x_2, x_3], & \beta_1 = -[x_2, \zeta_3] - [\zeta_2, x_3], & \gamma_1 = [\zeta_2, \zeta_3], \\ \alpha_2 = [x_3, x_1], & \beta_2 = -[x_3, \zeta_1] - [\zeta_3, x_1], & \gamma_2 = [\zeta_3, \zeta_1], \\ \alpha_3 = [x_1, x_2], & \beta_3 = -[x_1, \zeta_2] - [\zeta_1, x_2], & \gamma_3 = [\zeta_1, \zeta_2]. \end{cases}$$

* *Vector notation.* If a: a_1, a_2, a_3, and b: b_1, b_2, b_3 are ordered triples of numbers, their inner and outer products shall be denoted by $(a \mid b)$ and $\widehat{a\,b}$ respectively:

$$(a \mid b) = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

$$\widehat{a\,b}: \; a_2 b_3 - a_3 b_2,\; a_3 b_1 - a_1 b_3,\; a_1 b_2 - a_2 b_1.$$

Hence, α, β, γ are relatively invariant vectors of weight one, except that, in the case of β, the signs may change.

The vectors α, β, γ stand in close relationship, respectively, to the three fundamental differential forms of S. We write these forms, with their discriminants, as follows:

$$A = (dx \mid dx) = Edu^2 + 2Fdudv + Gdv^2, \quad D^2 = EG - F^2,$$
$$B = -(dx \mid d\xi) = edu^2 + 2fdudv + gdv^2, \quad d^2 = eg - f^2,$$
$$C = (d\xi \mid d\xi) = \mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2, \quad \mathcal{D}^2 = \mathcal{E}\mathcal{G} - \mathcal{F}^2.$$

The inner product of two of the vectors α, β, γ, or of one of these vectors with itself, is relatively invariant with respect to a change of parameters, and absolutely invariant with respect to rigid motions. The six invariants obtained in this way are essentially the fundamental algebraic invariants of the three quadratic forms with respect to changes of parameters:

	Form	*Discriminant*	*Simultaneous Invariant*
	A	$D^2 = (\alpha \mid \alpha)$,	$I(A, B) = (\alpha \mid \beta)$,
(2)	B	$D^2 = (\alpha \mid \gamma)$,	$I(A, C) = (\beta \mid \beta) - 2(\alpha \mid \gamma)$,
	C	$D^2 = (\gamma \mid \gamma)$,	$I(B, C) = (\beta \mid \gamma)$.

Since α, β, γ are all normal to S, and α is never a null vector, γ and β are scalar multiples of α. The multipliers are respectively K and K', the total and mean curvatures of the surface:

$$\gamma = K\alpha, \quad \beta = K'\alpha.$$

When these values of γ and β are substituted into (2), the six invariants in (2) become functions of $(\alpha \mid \alpha)$, K, K'. Hence, all the algebraic relative invariants of the three fundamental forms with respect to changes of parameters are functions of $(\alpha \mid \alpha)$, K, K'. But $(\alpha \mid \alpha)$, K, K' form a complete system of absolute invariants, with respect to rigid motions, of the three vectors α, β, γ. Hence, the system of absolute invariants with respect to rigid motions of these three vectors, which are themselves relatively invariant with respect to changes of parameters, is identical with the system of algebraic relative invariants of the three fundamental forms with respect to changes of parameters.

3. *Invariant Vectors Tangent to the Surface.* If $\phi(u, v)$ is a function which is absolutely invariant with respect to a change of parameters, there is associated with it the relatively invariant vector

$$[x, \phi]: \qquad [x_1, \phi],\ [x_2, \phi],\ [x_3, \phi],$$

and also the vector

$$\Phi = \frac{1}{D}[x, \phi] = \frac{1}{D}(\phi_v x_u - \phi_u x_v), \tag{3}$$

which is absolutely invariant except perhaps for a change of signs.

If ϕ is not a constant, the equation $\phi(u, v) = \text{const.}$ represents on S a family of curves, and the vector Φ at a point P of S lies on the tangent, at P, to the curve of the family which passes through P. We agree to direct the curve so that the positive direction is that of Φ.

Suppose that $\psi(u, v) = \text{const.}$ represents a second family of curves on S and let

$$\Psi = \frac{1}{D}[x, \psi]$$

be the corresponding absolutely invariant vector.

The inner product of one of the vectors Φ, Ψ with itself, or with the other, is evidently an absolute invariant. The invariants thus obtained are simply the familiar differential parameters of the first order:

$$\Delta_1\phi = (\Phi \mid \Phi), \qquad \Delta_1(\phi, \psi) = (\Phi \mid \Psi).$$

We have, also,

$$\Theta_1^2(\phi, \psi) = (\tilde{\Phi}\tilde{\Psi} \mid \tilde{\Phi}\tilde{\Psi}),$$

where

$$\Theta_1(\phi, \psi) = \frac{1}{D}[\phi, \psi].$$

The usual formulas for the angle ω from the directed curve $\phi = \text{const.}$ to the directed curve $\psi = \text{const.}$, namely,

$$\cos\omega = \frac{\Delta_1(\phi, \psi)}{(\Delta_1\phi)^{1/2}(\Delta_1\psi)^{1/2}}, \qquad \sin\omega = \frac{\Theta_1(\phi, \psi)}{(\Delta_1\phi)^{1/2}(\Delta_1\psi)^{1/2}},$$

now reduce to the obvious forms

$$\cos\omega = \frac{(\Phi \mid \Phi)}{(\Phi \mid \Phi)^{1/2}(\Psi \mid \Psi)^{1/2}}, \qquad \sin\omega = \frac{\Theta_1(\phi, \psi)}{(\Phi \mid \Phi)^{1/2}(\Psi \mid \Psi)^{1/2}}.$$

Since the length of the vector Φ is $(\Delta_1\phi)^{1/2}$, the vector $\zeta\Phi$ gives, in direction and length, the directional derivative of $\phi(u, v)$ of maximum value. It is the generalization of the gradient of a function in the plane.

The unit vector, $\partial x/\partial s$, in the direction of the vector Φ is

$$\frac{\partial x}{\partial s} = \bar{\Phi} = \frac{\Phi}{(\Phi \mid \Phi)^{1/2}} = \frac{1}{D}\frac{[x, \phi]}{(\Delta_1\phi)^{1/2}}. \tag{4}$$

Hence, the rate of change of an arbitrary function $\chi(u, v)$ in the positive direction of the curve $\phi(u, v) = \text{const.}$ is

$$\frac{\partial\chi}{\partial s} = \frac{1}{D}\frac{[\chi, \phi]}{(\Delta_1\phi)^{1/2}}. \tag{5}$$

4. *Differential Parameters of the Second Order. Geodesic Curvature.* Jacobians of invariant functions may be employed to form scalar invariants as well as invariant vectors.

For example, the differential parameter $\Delta_2\phi$, expressed in terms of Jacobians in a form which is obviously invariant with respect to a change of parameters, is

$$\Delta_2\phi = \frac{1}{D}\sum_{i=1}^{3}[x_i, \Phi_i]. \tag{6}$$

Equivalent to this is the expression

$$\Delta_2\phi = \frac{1}{D}\left[\frac{\partial}{\partial v}(x_u \mid \Phi) - \frac{\partial}{\partial u}(x_v \mid \Phi)\right],$$

which, when for Φ is substituted the value given by (3), expands into the usual form.

Again, the geodesic curvature of a curve $\phi(u, v) = \text{const.}$ is given by

$$\frac{1}{\rho} = -\frac{1}{D}\sum_{i=1}^{3}[x_i, \bar{\Phi}_i]. \tag{7}$$

For this relation may be put into the form

$$\text{(7a)} \qquad \frac{1}{\rho} = \frac{1}{D}\left[\frac{\partial}{\partial u}\left(x_v \left| \frac{\partial x}{\partial s}\right.\right) - \frac{\partial}{\partial v}\left(x_u \left| \frac{\partial x}{\partial s}\right.\right)\right],$$

and may then be expanded into the classical formula of Bonnet.

B. Directional Derivatives

5. *Condition of Integrability.* Let there be given a system of curves on S consisting of the family of curves $C:\phi(u, v) = \text{const.}$, and the family of curves $C':\psi(u, v) = \text{const.}$ Assume that $[\phi, \psi] > 0$, that is, that the curves of the families are so directed that $0 < \omega < \pi$, where ω is the angle from the directed curve C to the directed curve C', measured in the positive sense.

The unit vectors, ξ and ξ', in the positive directions of the curves C and C', are, by (4),

$$\text{(8)} \qquad \begin{aligned} \xi &= \frac{\partial x}{\partial s} = \frac{1}{D}\left[\frac{\phi_v}{(\Delta_1\phi)^{1/2}} x_u - \frac{\phi_u}{(\Delta_1\phi)^{1/2}} x_v\right], \\ \xi' &= \frac{\partial x}{\partial s'} = \frac{1}{D}\left[\frac{\psi_v}{(\Delta_1\psi)^{1/2}} x_u - \frac{\psi_u}{(\Delta_1\psi)^{1/2}} x_v\right]. \end{aligned}$$

Hence

$$x_u \sin\omega = -\frac{\psi_u}{(\Delta_1\psi)^{1/2}}\xi + \frac{\phi_u}{(\Delta_1\phi)^{1/2}}\xi',$$

$$x_v \sin\omega = -\frac{\psi_v}{(\Delta_1\psi)^{1/2}}\xi + \frac{\phi_v}{(\Delta_1\psi)^{1/2}}\xi',$$

and

$$dx \sin\omega = -\frac{d\psi}{(\Delta_1\psi)^{1/2}}\xi + \frac{d\phi}{(\Delta_1\phi)^{1/2}}\xi'.$$

Thus, the linear element of the surface, referred to the system of curves C and C', is

$$\text{(9)} \quad \sin^2\omega\,(dx \mid dx) = \frac{d\psi^2}{\Delta_1\psi} - 2\cos\omega\,\frac{d\psi}{(\Delta_1\psi)^{1/2}}\,\frac{d\phi}{(\Delta_1\phi)^{1/2}} + \frac{d\phi^2}{\Delta_1\phi}.$$

The differentials of arc, ds and ds', of the curves C and C', are, then, respectively,

$$\text{(10)} \qquad ds = -\frac{d\psi}{\sin\omega\,(\Delta_1\psi)^{1/2}}, \qquad ds' = +\frac{d\phi}{\sin\omega\,(\Delta_1\phi)^{1/2}}.$$

The arcs are measured in the positive directions of the curves, and the signs in (10) result from the fact that the functions ϕ and ψ increase in the directions which lie to the left of the curves C and C', when these curves, looked at from the side of S in the positive direction of the normal ζ, are traced in their positive senses.

The directional derivatives, $\partial\chi/\partial s$ and $\partial\chi/\partial s'$, of an arbitrary function $\chi(u, v)$ in the positive directions of the curves C and C', are, by (5),

$$\frac{\partial\chi}{\partial s} = \frac{1}{D}\left[\frac{\phi_v}{(\Delta_1\phi)^{1/2}}\chi_u - \frac{\phi_u}{(\Delta_1\phi)^{1/2}}\chi_v\right], \tag{11}$$

$$\frac{\partial\chi}{\partial s'} = \frac{1}{D}\left[\frac{\psi_v}{(\Delta_1\psi)^{1/2}}\chi_u - \frac{\psi_u}{(\Delta_1\psi)^{1/2}}\chi_v\right].$$

Solving these equations for χ_u and χ_v, we have

$$\chi_u = -\frac{\psi_u}{\sin\omega\,(\Delta_1\psi)^{1/2}}\frac{\partial\chi}{\partial s} + \frac{\phi_u}{\sin\omega\,(\Delta_1\phi)^{1/2}}\frac{\partial\chi}{\partial s'}, \tag{12}$$

$$\chi_v = -\frac{\psi_v}{\sin\omega\,(\Delta_1\psi)^{1/2}}\frac{\partial\chi}{\partial s} + \frac{\phi_v}{\sin\omega\,(\Delta_1\phi)^{1/2}}\frac{\partial\chi}{\partial s'}.$$

Forming $d\chi$ from these relations and introducing ds and ds' from (10), we find

$$d\chi = \frac{\partial\chi}{\partial s}ds + \frac{\partial\chi}{\partial s'}ds'. \tag{13}$$

On the other hand, if we differentiate the first of the equations (12) with respect to v and the second with respect to u and equate the resulting expressions, the equation obtained reduces to

$$\frac{\partial}{\partial s'}\frac{\partial\chi}{\partial s} - \frac{\partial}{\partial s}\frac{\partial\chi}{\partial s'} = \frac{1}{\sigma}\frac{\partial\chi}{\partial s} + \frac{1}{\sigma'}\frac{\partial\chi}{\partial s'}, \tag{14}$$

where

$$\frac{1}{\sigma} = \frac{\partial}{\partial s'}\log\left(\sin\omega\,(\Delta_1\psi)^{1/2}\right), \quad \frac{1}{\sigma'} = -\frac{\partial}{\partial s}\log\left(\sin\omega\,(\Delta_1\phi)^{1/2}\right). \tag{15}$$

A necessary and sufficient condition that two functions of u and v, which we denote by $\partial\chi/\partial s$ and $\partial\chi/\partial s'$, be the directional derivatives

in the directions of the curves C and C′ of a function $\chi(u, v)$ *is that*

$$\frac{\partial}{\partial s'}\frac{\partial \chi}{\partial s} - \frac{\partial}{\partial s}\frac{\partial \chi}{\partial s'} = \frac{1}{\sigma}\frac{\partial \chi}{\partial s} + \frac{1}{\sigma'}\frac{\partial \chi}{\partial s'}. \tag{14}$$

When the condition is satisfied,

$$\chi = \int \frac{\partial \chi}{\partial s} ds + \frac{\partial \chi}{\partial s'} ds'.$$

6. *The Invariants* $1/\sigma$, $1/\sigma'$. In seeking the geometric interpretations of the invariants $1/\sigma$ and $1/\sigma'$ given by (15), we consider first the significance of their vanishing.*

The curves C′ are equidistant curves with respect to the curves C, that is, every two curves C′ cut segments of equal length from the curves C, if and only if the invariant $1/\sigma$ *is zero.*

For, if $1/\sigma = 0$, the function $\sin \omega\, (\Delta_1 \psi)^{1/2}$ is constant along each curve C' and is, therefore, a function of ψ. Then, ds, as given by (10), is actually the exact differential of a function $s(\psi)$, which is the common arc of the curves C measured from a fixed curve C' to an arbitrary curve C'. The argument may be reversed.

The geometric interpretation of

$$\frac{1}{\sigma} = \frac{\partial}{\partial s'} \log\, (\sin \omega\, (\Delta_1 \psi)^{1/2})$$

is now clear. It is a measure of the deviation from one another of the curves C' with respect to the curves C. When we think of $d\psi$ in (10) as having been given a fixed value, it is precisely, except for sign, the logarithmic rate of change, in the direction of the general curve C', of the element of arc of the general curve C. We shall call it *the relative variation of the element of arc of the curve C with respect to the curve C′.*†

The relative variation, $1/\sigma$, of the element of arc of C with respect to C' is taken in the positive direction of C', that is, in the direction on C' to the left of C in the sense described in §5.

* The significance of the vanishing of $1/\sigma$ and $1/\sigma'$ is well known; see Lilienthal, *Vorlesungen*, vol. 2, p. 228.

† It is to be noted that $1/\sigma$ is unchanged when ψ is replaced by $F(\psi)$. This is not true, for example, of $(\partial/\partial s')(\sin \omega(\Delta_1 \psi)^{1/2})$.

Accordingly, we should take the relative variation, $1/\sigma'$, of the element of arc of C' with respect to C in the direction on C to the left of C'. This is the negative direction on C. Hence, we should have, and, by (15), actually do have

$$\frac{1}{\sigma'} = -\frac{\partial}{\partial s}\log\left(\sin\omega\,(\Delta_1\phi)^{1/2}\right).$$

From the law of change, with the direction, of the directional derivative, it is evident that

$$\sin\omega(\Delta_1\psi)^{1/2} = -\frac{\partial\psi}{\partial s}, \quad \sin\omega\,(\Delta_1\phi)^{1/2} = \frac{\partial\phi}{\partial s'}.$$

Hence, we get the formulas

$$\frac{1}{\sigma} = \frac{\partial}{\partial s'}\log\left(-\frac{\partial\psi}{\partial s}\right), \quad \frac{1}{\sigma'} = -\frac{\partial}{\partial s}\log\left(\frac{\partial\psi}{\partial s'}\right),$$

which give new interpretations of $1/\sigma$ and $1/\sigma'$.

We suppose now that the curves C and C' form an orthogonal system: $\omega = \pi/2$, and compute their geodesic curvatures, $1/\rho$ and $1/\rho'$. Substituting for x_u and x_v in (7a) their values in terms of ξ and ξ', and setting, for $\partial x/\partial s$, ξ and ξ' in turn, we find

$$(16)\qquad \begin{aligned} \frac{1}{\rho} &= \frac{1}{D}\left[\frac{\partial}{\partial u}\left(-\frac{\psi_v}{(\Delta_1\psi)^{1/2}}\right) - \frac{\partial}{\partial v}\left(-\frac{\psi_u}{(\Delta_1\psi)^{1/2}}\right)\right], \\ \frac{1}{\rho'} &= \frac{1}{D}\left[\frac{\partial}{\partial u}\left(\frac{\phi_v}{(\Delta_1\phi)^{1/2}}\right) - \frac{\partial}{\partial v}\left(\frac{\phi_u}{(\Delta_1\phi)^{1/2}}\right)\right]. \end{aligned}$$

Hence, we have

$$(16a)\qquad \frac{1}{\rho} = \frac{\partial}{\partial s'}\log(\Delta_1\psi)^{1/2} = \frac{1}{\sigma}, \quad \frac{1}{\rho'} = -\frac{\partial}{\partial s}\log(\Delta_1\phi)^{1/2} = \frac{1}{\sigma'}.$$

If the curves C and C' form an orthogonal system, the relative variations of their elements of arc, $1/\sigma$ and $1/\sigma'$, are equal respectively to their geodesic curvatures, $1/\rho$ and $1/\rho'$.

We thus have new interpretations of the geodesic curvatures of the curves of an orthogonal system.

7. *The Modified Directional Derivatives.* The condition of integrability (14) tells us that the order of the operations repre-

sented by the symbols $\partial/\partial s$ and $\partial/\partial s'$ cannot be inverted. We propose to introduce modifications of the directional derivatives $\partial/\partial s$, $\partial/\partial s'$, which, when used in conjunction with these derivatives, will not only remedy this defect, but also render our methods more powerful and elegant.

The modifications of $\partial/\partial s$ and $\partial/\partial s'$, which we shall denote by $\nabla/\nabla s$ and $\nabla/\nabla s'$, are defined as follows:*

$$\frac{\nabla\chi}{\nabla s} = \frac{\partial\chi}{\partial s} + \frac{1}{\sigma'}\chi, \quad \frac{\nabla\chi}{\nabla s'} = \frac{\partial\chi}{\partial s'} - \frac{1}{\sigma}\chi. \tag{17}$$

According to the definition,

$$\frac{\nabla}{\nabla s'}\frac{\partial\chi}{\partial s} = \frac{\partial}{\partial s'}\frac{\partial\chi}{\partial s} - \frac{1}{\sigma}\frac{\partial\chi}{\partial s}, \qquad \frac{\nabla}{\nabla s}\frac{\partial\chi}{\partial s'} = \frac{\partial}{\partial s}\frac{\partial\chi}{\partial s'} + \frac{1}{\sigma'}\frac{\partial\chi}{\partial s'}.$$

Hence, the condition of integrability (14) becomes

$$\frac{\nabla}{\nabla s'}\frac{\partial\chi}{\partial s} = \frac{\nabla}{\nabla s}\frac{\partial\chi}{\partial s'}. \tag{18}$$

A necessary and sufficient condition that $Q(u, v)ds + Q'(u, v)ds'$ be an exact differential is that

$$\frac{\nabla Q}{\nabla s'} = \frac{\nabla Q'}{\nabla s}.$$

The modified derivative of a sum is the sum of the modified derivatives. But the standard law for differentiating a product does not hold for the modified differentiation. We have, instead,

$$\frac{\nabla(\chi_1\chi_2)}{\nabla s} = \chi_1\frac{\nabla\chi_2}{\nabla s} + \chi_2\frac{\partial\chi_1}{\partial s} = \chi_1\frac{\partial\chi_2}{\partial s} + \chi_2\frac{\nabla\chi_1}{\nabla s}.$$

The fact that one of the factors is subjected to ordinary differentiation, and the other to the modified differentiation, works out advantageously in practice, as will be seen later.

When the given system of curves is an orthogonal system, $1/\sigma = 1/\rho$ and $1/\sigma' = 1/\rho'$. Consequently, in this case, both

* Though the author came upon it otherwise, the reader will find the germ of the idea of the modified directional derivative in the abbreviations $(\partial/\partial s + 1/\rho')\chi$ and $(\partial/\partial s' - 1/\rho)\chi$ employed by Cesàro, loc. cit., in the discussion of the case in which the curves C and C' form an orthogonal system.

modified directional derivatives are identical with the corresponding ordinary derivatives if and only if S is a plane and toe system of curves consists of two orthogonal pencils of straight lines.

On the other hand, in the general case, both modified derivatives coincide with the corresponding ordinary derivatives for every system of curves which clothes S in the sense of Tchebycheff. For $1/\sigma$ and $1/\sigma'$ both vanish if and only if the curves of each family of the system are equidistant with respect to the curves of the other family, and this is a characteristic property of a system of curves which clothes the surface.

C. The Canonical Differential Equations of the Families of Curves of a System

8. *The Canonical Differential Equations, Derived from Finite Equations.* In connection with the system of curves consisting of the family of curves $C:\phi(u, v)=\text{const.}$ and the family of curves $C':\psi(u, v)=\text{const.}$, we recall the expressions

$$(10) \qquad ds = -\frac{d\psi}{\sin\omega\,(\Delta_1\psi)^{1/2}}, \quad ds' = \frac{d\phi}{\sin\omega\,(\Delta_1\phi)^{1/2}}$$

for the differentials of arc of the curves C and C'.

When we set

$$(19) \qquad \begin{aligned} A' &= -\frac{\psi_u}{\sin\omega(\Delta_1\psi)^{1/2}}, \quad B' = -\frac{\psi_v}{\sin\omega(\Delta_1\psi)^{1/2}}, \\ A &= \frac{\phi_u}{\sin\omega(\Delta_1\phi)^{1/2}}, \quad B = \frac{\phi_v}{\sin\omega(\Delta_1\phi)^{1/2}}, \end{aligned}$$

these expressions become

$$(20) \qquad ds = A'du + B'dv, \quad ds' = Adu + Bdv.$$

Since ϕ, ψ, $\Delta_1\phi$, and $\Delta_1\psi$ are absolute invariants with respect to a change of parameters, and $\sin\omega$ is an absolute invariant except perhaps for a change of signs, the linear differential forms $Adu+Bdv$ and $A'du+B'dv$ are absolute invariants except perhaps for a change of signs.

It is evident that

$$Adu + Bdv = 0, \qquad A'du + B'dv = 0$$

are differential equations, respectively, of the families of curves C and C'. We shall call them *the canonical differential equations* of these families and employ them exclusively, from now on, to represent the families.

The presence of the factor $\sin\omega$ in formulas (10) shows that a canonical differential equation of a family of curves C is determined only when there is associated with the family C a second family C'. In other words, the concept of canonical differential equations applies only to two families of curves forming a system, and not to individual families of curves.

We proceed now to express the theory thus far developed in terms of canonical differential equations and the coefficients A, B, A', B' contained in them.

From (9) we have, as the new form of the linear element,

$$(dx|dx) = ds^2 + 2\cos\omega\, dsds' + ds'^2. \tag{21}$$

According to (8),

$$\xi = \frac{\sin\omega}{D}(Bx_u - Ax_v), \qquad \xi' = -\frac{\sin\omega}{D}(B'x_u - A'x_v).$$

Substituting these expressions into

$$\overbrace{\xi\xi'} = \zeta\sin\omega,$$

we find

$$D = \sin\omega(A'B - AB').$$

Hence, when we set

$$\bar{D} = A'B - AB', \tag{22}$$

we have

$$D = \bar{D}\sin\omega.$$

The expressions for ξ and ξ' now become

$$\xi = \frac{1}{\bar{D}}(Bx_u - Ax_v), \qquad \xi' = -\frac{1}{\bar{D}}(B'x_u - A'x_v). \tag{23}$$

Similarly, we have

$$\frac{\partial\chi}{\partial s} = \frac{1}{\bar{D}}(B\chi_u - A\chi_v), \qquad \frac{\partial\chi}{\partial s'} = -\frac{1}{\bar{D}}(B'\chi_u - A'\chi_v). \tag{24}$$

The condition of integrability is of the same form as before,

where now the values of $1/\sigma$ and $1/\sigma'$ are given by the much simpler expressions

$$\frac{1}{\sigma} = \frac{1}{\bar{D}}(B_u' - A_v'), \qquad \frac{1}{\sigma'} = \frac{1}{\bar{D}}(B_u - A_v). \tag{25}$$

From (24) and (25) it is readily seen that the modified directional derivatives have the values

$$\begin{aligned} \frac{\nabla\chi}{\nabla s} &= \frac{\partial\chi}{\partial s} + \frac{1}{\sigma'}\chi = \frac{1}{\bar{D}}[(B\chi)_u - (A\chi)_v], \\ \frac{\nabla\chi}{\nabla s'} &= \frac{\partial\chi}{\partial s'} - \frac{1}{\sigma}\chi = -\frac{1}{\bar{D}}[(B'\chi)_u - (A'\chi)_v]. \end{aligned} \tag{26}$$

Thus, the vanishing of $\nabla\chi/\nabla s$ is a condition necessary and sufficient that χ be an integrating factor of $Adu+Bdv$, and the vanishing of $\nabla\chi/\nabla s'$ is a necessary and sufficient condition that χ be an integrating factor of $A'du+B'dv$.*

9. *Derivation of Canonical Differential Equations from Arbitrary Differential Equations.* The canonical differential equations of the families of curves C and C' are readily found, as we have seen, when the families of curves are defined by finite equations. They may be found with equal ease, without quadratures, when the families of curves are defined by differential equations, chosen at random.

Let differential equations defining the curves C and C' be

$$C: Mdu + Ndv = 0, \qquad C': \quad M'du + N'dv = 0.$$

We may assume that $MN'-NM'>0$, since otherwise we could change the signs of, say, M' and N'. Let, then, $I(u, v)>0$ and $I'(u, v)>0$ denote positive integrating factors of $Mdu+Ndv$ and $M'du+N'dv$, respectively, and form

* The corresponding conditions in terms of the ordinary derivatives are, respectively,

$$\frac{\partial \log\chi}{\partial s} = -\frac{1}{\sigma'}, \quad \frac{\partial \log\chi}{\partial s'} = \frac{1}{\sigma},$$

and these conditions are in keeping with the geometric interpretations of $1/\sigma$ and $1/\sigma'$. In particular, when $1/\sigma$ and $1/\sigma'$ are both zero, the linear forms $A'du+B'dv$ and $Adu+Bdv$ are exact differentials, s and s' are parameters for the surface, and the linear element (21) becomes characteristic for a system of curves which clothes the surface.

$$(27) \qquad d\phi = IMdu + INdv, \quad d\psi = I'M'du + I'N'dv.$$

Then $[\phi, \psi] > 0$ and $\sin \omega > 0$.

The quantities A, B, A', B' may now be computed from (27), by means of (19), and, what is more important, the values obtained for them are independent of I and I'.

D. The Surface Referred to an Orthogonal System of Curves

10. *The Differential Equations of the Vectors of the Trihedral* ξ, ξ', ζ. Let the canonical differential equations of the families of curves C and C' of an orthogonal system* on S be, respectively,

$$Adu + Bdv = 0, \qquad A'du + B'dv = 0.$$

The important formulas for this case may be obtained from those of §8 by setting $\sin \omega = 1$, $\overline{D} = D$, and replacing $1/\sigma$ and $1/\sigma'$ by $1/\rho$ and $1/\rho'$.

Thus, the unit vectors ξ and ξ' in the positive directions of the curves C and C' are

$$(28) \qquad \xi = \frac{1}{D}(Bx_u - Ax_v), \quad \xi' = -\frac{1}{D}(B'x_u - A'x_v),$$

where

$$D = A'B - AB'.$$

Hence

$$(29) \qquad x_u = A'\xi + A\xi', \quad x_v = B'\xi + B\xi'.$$

For the geodesic curvatures of C and C' and the directional derivatives in their positive directions, we have

$$(30) \qquad \begin{aligned} \frac{1}{\rho} &= \frac{1}{D}(B_u' - A_v'), & \frac{1}{\rho'} &= \frac{1}{D}(B_u - A_v), \\ \frac{\partial \chi}{\partial s} &= \frac{1}{D}(B\chi_u - A\chi_v), & \frac{\partial \chi}{\partial s'} &= -\frac{1}{D}(B'\chi_u - A'\chi_v), \\ \frac{\nabla \chi}{\nabla s} &= \frac{1}{D}[(B\chi)_u - (A\chi)_v] = \frac{\partial \chi}{\partial s} + \frac{1}{\rho'}\chi, \\ \frac{\nabla \chi}{\nabla s'} &= -\frac{1}{D}[(B'\chi)_u - (A'\chi)_v] = \frac{\partial \chi}{\partial s'} - \frac{1}{\rho}\chi, \end{aligned}$$

* For the further development of the treatment of the surface referred to an arbitrary system of curves, see author, loc. cit.

where

$$D = A'B - AB'.$$

We also have

$$d\chi = \frac{\partial \chi}{\partial s} ds + \frac{\partial \chi}{\partial s'} ds',$$

where

$$ds = A'du + B'dv, \quad ds' = Adu + Bdv.$$

The unit vectors ξ, ξ', in the positive directions of curves C, C', and the unit vector ζ normal to the surface form a rectangular trihedral which has the same disposition as the axes: $(\xi\xi'\zeta) = 1$. We propose to find expressions for the directional derivatives, in the directions of the curves C and C', of the vectors of this trihedral, and to write them as linear combinations of these vectors. Since ξ, ξ', ζ are mutually orthogonal unit vectors, the determinant of the coefficients in the linear combinations which are respectively the values of the directional derivatives of ξ, ξ', ζ in a specific direction is skew-symmetric.

We obtain the values of $\partial\xi/\partial s$, $\partial\xi'/\partial s$, $\partial\zeta/\partial s$ by considering an arbitrary curve C. The geodesic curvature of C is

$$\frac{1}{\rho} = \frac{(\eta \mid \xi')}{R}$$

where η is the unit vector in the positive direction of the principal normal to C, and $1/R$ is the curvature of C. But $\eta/R = \partial\xi/\partial s$, and hence

$$\frac{1}{\rho} = \left(\frac{\partial \xi}{\partial s}\middle| \xi'\right) = -\left(\frac{\partial \xi'}{\partial s}\middle| \xi\right), \tag{31}$$

where the second of the two values results from the fact that $(\xi \mid \xi') = 0$.

The normal curvature of C is given by the formula

$$\frac{1}{r} = -\frac{(dx \mid d\zeta)}{(dx \mid dx)} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}.$$

Since, along C, $(dx \mid dx) = ds^2$ and $\partial x/\partial s = \xi$,

$$\frac{1}{r} = -\left(\frac{\partial \zeta}{\partial s}\middle| \xi\right) = \left(\frac{\partial \xi}{\partial s}\middle| \zeta\right). \tag{32}$$

The familiar formula,

$$\frac{1}{\tau} = -\frac{(Ef - Fe)du^2 + (Eg - Ge)dudv + (Fg - Gf)dv^2}{D(Edu^2 + 2Fdudv + Gdv^2)},$$

for the geodesic curvature of C amounts to nothing more than

$$\frac{1}{\tau} = -\left(\frac{\partial\zeta}{\partial s}\frac{\partial x}{\partial s}\zeta\right).$$

But $\partial x/\partial s = \xi$ and $\overline{\xi\zeta} = -\xi'$. Hence

$$\frac{1}{\tau} = \left(\frac{\partial\zeta}{\partial s}\middle|\,\xi'\right) = -\left(\frac{\partial\xi'}{\partial s}\middle|\,\zeta\right). \tag{33}$$

Formulas (31), (32), (33) are equivalent to

$$\begin{aligned}
\frac{\partial\xi}{\partial s} &= \frac{1}{\rho}\xi' + \frac{1}{r}\zeta,\\
\frac{\partial\xi'}{\partial s} &= -\frac{1}{\rho}\xi - \frac{1}{\tau}\zeta,\\
\frac{\partial\zeta}{\partial s} &= -\frac{1}{r}\xi + \frac{1}{\tau}\xi'.
\end{aligned} \tag{34}$$

The trihedral for the general curve C' which corresponds to the trihedral ξ, ξ', ζ for the curve C is $\xi', \xi'' = -\xi, \zeta$. Hence,

$$\begin{aligned}
\frac{\partial\xi'}{\partial s'} &= -\frac{1}{\rho'}\xi + \frac{1}{r'}\zeta,\\
\frac{\partial\xi}{\partial s'} &= \frac{1}{\rho'}\xi' + \frac{1}{\tau'}\zeta,\\
\frac{\partial\zeta}{\partial s} &= \frac{1}{r'}\xi' - \frac{1}{\tau'}\xi
\end{aligned} \tag{35}$$

where $1/\rho'$, $1/r'$, $1/\tau'$ are respectively the geodesic curvature, the normal curvature, and the geodesic torsion of the curve C'.

Consider, now, the condition of integrability (14) of the equations

$$\frac{\partial x}{\partial s} = \xi, \qquad \frac{\partial x}{\partial s'} = \xi', \tag{36}$$

namely,

$$\frac{\partial \xi}{\partial s'} - \frac{\partial \xi'}{\partial s} = \frac{1}{\rho}\xi + \frac{1}{\rho'}\xi'. \tag{37}$$

When for $\partial\xi/\partial s'$ and $\partial\xi'/\partial s$ are substituted their values from (34) and (35), this condition reduces to

$$\frac{1}{\tau'} = -\frac{1}{\tau}.$$

Thus, *the required system of differential equations for the vectors* ξ, ξ', ζ *is*

$$\begin{aligned}
&\frac{\partial \xi}{\partial s} = \frac{1}{\rho}\xi' + \frac{1}{r}\zeta, \quad \frac{\partial \xi}{\partial s'} = \frac{1}{\rho'}\xi' - \frac{1}{\tau}\zeta,\\
&\frac{\partial \xi'}{\partial s} = -\frac{1}{\rho}\xi - \frac{1}{\tau}\zeta, \quad \frac{\partial \xi'}{\partial s'} = -\frac{1}{\rho'}\xi + \frac{1}{r'}\zeta,\\
&\frac{\partial \zeta}{\partial s} = -\frac{1}{r}\xi + \frac{1}{\tau}\xi', \quad \frac{\partial \zeta}{\partial s'} = \frac{1}{\tau}\xi - \frac{1}{r'}\xi'.
\end{aligned} \tag{38}$$

We know that

$$dx = \xi ds + \xi' ds'.$$

By the last pair of equations in (38),

$$-d\zeta = \left(\frac{1}{r}\xi - \frac{1}{\tau}\xi'\right)ds + \left(-\frac{1}{\tau}\xi + \frac{1}{r'}\xi'\right)ds'.$$

Hence

$$\begin{aligned}
(dx \mid dx) &= ds^2 + ds'^2,\\
-(dx \mid d\zeta) &= \frac{1}{r}ds^2 - \frac{2}{\tau}ds ds' + \frac{1}{r'}ds'^2,\\
(d\zeta \mid d\zeta) &= \left(\frac{1}{r^2} + \frac{1}{\tau^2}\right)ds^2 - \frac{2}{\tau}K' ds ds' + \left(\frac{1}{r'^2} + \frac{1}{\tau'^2}\right)ds'^2.
\end{aligned} \tag{39}$$

The three fundamental quadratic forms of the surface are expressible in terms of (a) *the canonical linear forms,* $ds' = Adu + Bdv$ *and* $ds = A'du + B'dv$, *and* (b) *the normal curvatures,* $1/r$, $1/r'$, *and geodesic torsion,* $1/\tau$, *associated with the curves* C *and* C'.

It is evident that we have now all the advantages of having the surface referred to the given orthogonal system of curves, while the parametric curves themselves remain arbitrary.

11. *The Fundamental Theorem.* The relation (37), which expresses the fact that equations (36) are compatible, has already been taken account of in equations (38).

If we apply the condition of integrability, in the form

$$\frac{\nabla}{\nabla s'}\frac{\partial \chi}{\partial s} = \frac{\nabla}{\nabla s}\frac{\partial \chi}{\partial s'},$$

to the directional derivatives of ξ, ξ', ζ in (38), we obtain the following relations:*

$$(40) \qquad \frac{\nabla}{\nabla s'}\left(\frac{1}{\rho}\right) - \frac{\nabla}{\nabla s}\left(\frac{1}{\rho'}\right) = K, \quad \text{where} \quad K = \frac{1}{rr'} - \frac{1}{\tau^2},$$

$$(41) \qquad \begin{aligned} &\frac{\nabla}{\nabla s'}\left(\frac{1}{r}\right) + \frac{\nabla}{\nabla s}\left(\frac{1}{\tau}\right) + \frac{1}{\rho}\frac{1}{r'} + \frac{1}{\rho'}\frac{1}{\tau} = 0, \\ &\frac{\nabla}{\nabla s}\left(\frac{1}{r'}\right) + \frac{\nabla}{\nabla s'}\left(\frac{1}{\tau}\right) - \frac{1}{\rho'}\frac{1}{r} - \frac{1}{\rho}\frac{1}{\tau} = 0. \end{aligned}$$

Equation (40) is the Gauss equation, expressed invariantly with reference to the given orthogonal system, and equations (41) are the equations of Codazzi, in similar form.

It is clear from (30) that equations (40) and (41) are equations which connect the seven fundamental quantities, A, B, A', B', $1/r$, $1/r'$, $1/\tau$, and their partial derivatives with respect to u and v. They constitute necessary conditions for the existence of a surface $x=x(u, v)$ for which the equations $Adu+Bdv=0$ and $A'du+B'dv=0$ define an orthogonal system of curves C and C' whose normal curvatures and geodesic torsions are $1/r$, $1/r'$ and $1/\tau$, $-1/\tau$.

* In applying the condition, we have to find the modified derivatives of products of the form $k\delta$, where k is a function and δ is one of the vectors ξ, ξ', ζ. To one factor of the product we must apply the ordinary differentiation, and to the other, the modified differentiation. The advantage of this is evident. For, we may take δ as the factor to which to apply the ordinary differentiation, and the ordinary derivatives of ξ, ξ', ζ are just what are given us in (38).

These conditions are also sufficient. For, if the seven quantities in question satisfy equations (40) and (41), the system of differential equations (38), subject to the conditions

$$(\xi\,|\,\xi) = (\xi'\,|\,\xi') = (\zeta\,|\,\zeta) = 1,\ (\xi\,|\,\xi') = (\xi\,|\,\zeta) = (\xi'\,|\,\zeta) = 0,\ (\xi\xi'\zeta) = 1,$$

is integrable. The resulting rectangular trihedral of unit vectors, ξ, ξ', ζ, is determined to within a rigid motion.

When the expressions found for ξ and ξ' are substituted in equations (36), these equations are integrable. A surface,

$$x = \int \xi ds + \xi' ds',$$

is thus determined to within its position in space. Since the three fundamental forms of this surface are as given in (39), it is readily shown to have the desired properties.

If two linear differential forms, $Adu+Bdv$ and $A'du+B'dv$, and three functions, $1/r$, $1/r'$, and $1/\tau$, are given, such that equations (40) *and* (41), *where*

$$\frac{1}{\rho} = \frac{1}{D}(B_u' - A_v'), \qquad \frac{1}{\rho'} = \frac{1}{D}(B_u - A_v),$$

$$\frac{\nabla\chi}{\nabla s} = \frac{1}{D}[(B\chi)_u - (A\chi)_v], \quad \frac{\nabla\chi}{\nabla s'} = -\frac{1}{D}[(B'\chi)_u - (A'\chi)_v],$$

and

$$D = A'B - AB' \neq 0,$$

are satisfied, there exists a surface, unique to within its position in space, on which the equations $Adu+Bdv=0$, $A'du+B'dv=0$ define two orthogonal families of curves which have $1/r$, $1/r'$ as their normal curvatures, and $1/\tau$, $-1/\tau$ as their geodesic torsions.

We permit ourselves, at this point, one application. The equation $K=0$, that is,

$$\frac{\nabla}{\nabla s'}\left(\frac{1}{\rho}\right) = \frac{\nabla}{\nabla s}\left(\frac{1}{\rho'}\right),$$

says that the expression

$$\frac{ds}{\rho} + \frac{ds'}{\rho'}$$

is an exact differential. Thus, there exists a function, $\theta(u, v)$, for which

$$\frac{\partial\theta}{\partial s} = \frac{1}{\rho}, \qquad \frac{\partial\theta}{\partial s'} = \frac{1}{\rho'}.$$

But these are precisely the conditions that the expressions

$$e^{\theta i}[(A - iA')du + (B - iB')dv],$$
$$e^{-\theta i}[(A + iA')du + (B + iB')dv],$$

be exact differentials, as can be immediately verified. The product of the two expressions can, therefore, be written in the form $du_1^2 + dv_1^2$. But this product is the linear element of the surface.

Thus, we have established, in a simple and elegant manner, a theorem whose proof by the usual methods is decidedly laborious, namely, the theorem to the effect that the linear element of a developable surface can be reduced by quadratures to the form $du^2 + dv^2$.

12. *The Surface Referred to the Lines of Curvature.* We shall denote the curves of the two families of lines of curvature by C_1 and C_2, the canonical differential equations which represent them by

$$A_1du + B_1dv = 0, \qquad A_2du + B_2dv = 0,$$

and the unit vectors in their positive directions, by $\xi^{(1)}$, $\xi^{(2)}$.

The following formulas are, then, self-explanatory:

$$D = A_2B_1 - A_1B_2,$$

$$\frac{1}{\rho_1} = \frac{1}{D}\left(\frac{\partial B_2}{\partial u} - \frac{\partial A_2}{\partial v}\right), \qquad \frac{1}{\rho_2} = \frac{1}{D}\left(\frac{\partial B_1}{\partial u} - \frac{\partial A_1}{\partial v}\right),$$

$$(42)\quad \frac{\partial\chi}{\partial s_1} = \frac{1}{D}(B_1\chi_u - A_1\chi_v), \qquad \frac{\partial\chi}{\partial s_2} = -\frac{1}{D}(B_2\chi_u - A_2\chi_v),$$

$$\frac{\nabla\chi}{\nabla s_1} = \frac{1}{D}[(B_1\chi)_u - (A_1\chi)_v], \quad \frac{\nabla\chi}{\nabla s_2} = -\frac{1}{D}[(B_2\chi)_u - (A_2\chi)_v].$$

Since the lines of curvature are characterized by the vanishing

of the geodesic torsion, the fundamental differential forms of S are, in this case,

$$ds_1^2 + ds_2^2, \quad \frac{1}{r_1}ds_1^2 + \frac{1}{r_2}ds_2^2, \quad \frac{1}{r_1^2}ds_1^2 + \frac{1}{r_2^2}ds_2^2,$$

and the fundamental differential equations,

$$\frac{\partial x}{\partial s_1} = \xi^{(1)}, \qquad \frac{\partial x}{\partial s_2} = \xi^{(2)},$$

$$(43)\quad \begin{aligned} \frac{\partial \xi^{(1)}}{\partial s_1} &= \frac{1}{\rho_1}\xi^{(2)} + \frac{1}{r_1}\zeta, & \frac{\partial \xi^{(1)}}{\partial s_2} &= \frac{1}{\rho_2}\xi^{(2)}, \\ \frac{\partial \xi^{(2)}}{\partial s_1} &= -\frac{1}{\rho_1}\xi^{(1)}, & \frac{\partial \xi^{(2)}}{\partial s_2} &= -\frac{1}{\rho_2}\xi^{(1)} + \frac{1}{r_2}\zeta, \\ \frac{\partial \zeta}{\partial s_1} &= -\frac{1}{r_1}\xi^{(1)}, & \frac{\partial \zeta}{\partial s_2} &= -\frac{1}{r_2}\xi^{(2)}. \end{aligned}$$

The Gauss-Codazzi equations take the simple forms

$$(44)\quad \begin{aligned} &\frac{\nabla}{\nabla s_2}\left(\frac{1}{\rho_1}\right) - \frac{\nabla}{\nabla s_1}\left(\frac{1}{\rho_2}\right) = K, \quad K = \frac{1}{r_1 r_2}, \\ &\frac{\nabla}{\nabla s_2}\left(\frac{1}{r_1}\right) + \frac{1}{\rho_1}\frac{1}{r_2} = 0, \qquad \frac{\nabla}{\nabla s_1}\left(\frac{1}{r_2}\right) - \frac{1}{\rho_2}\frac{1}{r_1} = 0. \end{aligned}$$

The fundamental theorem is also simpler, since it bears merely on the six functions A_1, B_1, A_2, B_2, $1/r_1$, $1/r_2$.

13. *A Complete System of Invariant Vectors.* We proceed to establish the following theorem, in which we refer to the unit vectors in the positive directions of the lines of curvature as *the principal vectors.*

Let $r_1\xi^{(1)}$, $r_2\xi^{(2)}$, ζ be three vectors whose components are functions of u and v, and suppose that $\xi^{(1)}$, $\xi^{(2)}$, ζ are unit vectors which form a rectangular trihedral having the same disposition as that of the axes. Form the quantities

$$A_1 = -r_2(\zeta_u \mid \xi^{(2)}), \quad B_1 = -r_2(\zeta_v \mid \xi^{(2)}),$$

$$A_2 = -r_1(\zeta_u \mid \xi^{(1)}), \quad B_2 = -r_1(\zeta_v \mid \xi^{(1)}),$$

$$D = A_2B_1 - A_1B_2 = r_1r_2(\xi^{(1)}\xi^{(2)}\zeta_u\zeta_v),$$

and substitute them in the expressions for $\partial/\partial s_1$, $\partial/\partial s_2$, $\nabla/\nabla s_1$, $\nabla/\nabla s_2$ *given in* (42). *Then, if the equation*

$$\frac{\Delta\xi^{(1)}}{\Delta s_2} = \frac{\Delta\xi^{(2)}}{\Delta s_1} \tag{45}$$

is satisfied, there exists a surface, uniquely determined to within a translation, whose normal vector is ζ *and whose principal vectors and principal radii of normal curvature are, respectively,* $\xi^{(1)}$, $\xi^{(2)}$, *and* r_1, r_2.*

Since equation (45) is satisfied, the equations

$$\frac{\partial x}{\partial s_1} = \xi^{(1)}, \quad \frac{\partial x}{\partial s_2} = \xi^{(2)}, \tag{46}$$

where $\partial/\partial s_1$, $\partial/\partial s_2$ are defined as in the theorem, are integrable. A surface $x = x(u, v)$ is thus determined to within a translation.

According to (46), the unit vector normal to this surface is ζ. Since, by the definitions of A_1, B_1, A_2, B_2, given in the theorem,

$$-\zeta_u = \frac{A_2}{r_1}\xi^{(1)} + \frac{A_1}{r_2}\xi^{(2)}, \quad -\zeta_v = \frac{B_2}{r_1}\xi^{(1)} + \frac{B_1}{r_2}\xi^{(2)},$$

or

$$\frac{\partial\zeta}{\partial s_1} = -\frac{1}{r_1}\xi^{(1)}, \quad \frac{\partial\zeta}{\partial s_2} = -\frac{1}{r_2}\xi^{(2)},$$

it follows that the surface has $\xi^{(1)}$, $\xi^{(2)}$ as its principal vectors and $1/r_1$, $1/r_2$ as its principal normal curvatures.

It follows from the theorem that $r_1\xi^{(1)}$, $r_2\xi^{(2)}$, ζ form a complete system of invariant vectors, or that the vectors $\xi^{(1)}$, $\xi^{(2)}$, ζ and the functions r_1, r_2 form a complete system of invariants, for a given surface.

A similar theorem may be stated in the case of an arbitrary orthogonal system. Here, it is the vectors ξ, ξ', ζ and the functions $1/r$, $1/r'$, $1/\tau$ which form the complete system of invariants.

14. *Integrating Factors.* From §8, we have the following result.

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* It is assumed that $r_1 r_2(\xi^{(1)}\xi^{(2)}\widehat{\zeta_u\zeta_v}) > 0$; in particular, the case of a developable surface is excluded.

Conditions necessary and sufficient that $m(u, v)$ and $n(u, v)$ be respectively integrating factors of $Adu+Bdv$ and $A'du+B'dv$ are

$$\frac{\nabla m}{\nabla s} = 0, \qquad \frac{\nabla n}{\nabla s'} = 0,$$

or

$$\frac{\partial \log m}{\partial s} = -\frac{1}{\rho'}, \qquad \frac{\partial \log n}{\partial s'} = \frac{1}{\rho}.$$

A corresponding theorem is valid in the case of the canonical linear differential forms A_1du+B_1dv and A_2du+B_2dv which, when equated to zero, define the lines of curvature.

These theorems form the basis for a systematic and comprehensive theory of integrating factors for the differential equations of families of curves on the surface. They enable us, not only to discuss the cases in which the families of curves can be found by quadratures, but also to exhibit the actual integrating factors.

We content ourselves with one example, that of an isometric system. If $Adu+Bdv$ and $A'du+B'dv$ have a common integrating factor, $m(u, v)$, then

$$\frac{\partial \log m}{\partial s} = -\frac{1}{\rho'}, \qquad \frac{\partial \log m}{\partial s'} = \frac{1}{\rho},$$

and conversely. But the condition of compatibility of these equations is

$$\frac{\nabla}{\nabla s}\left(\frac{1}{\rho}\right) + \frac{\nabla}{\nabla s'}\left(\frac{1}{\rho'}\right) = 0. \tag{47}$$

If this condition is fulfilled, $m(u, v)$, where

$$\log m = \int \frac{ds'}{\rho} - \frac{ds}{\rho'},$$

is a common integrating factor of $Adu+Bdv$ and $A'du+B'dv$, and the finite equations of the curves C and C' can be found by quadratures.

Furthermore, if we set

$$m(Adu + Bdv) = dv_1, \qquad m(A'du + B'dv) = du_1,$$

the linear element becomes

$$ds^2 = \frac{1}{m^2}(du_1^2 + dv_1^2),$$

where

$$m = \int \frac{dv_1}{\rho} - \frac{du_1}{\rho'}.$$

The condition (47) that the curves C and C' form an isometric system remains unaffected when ordinary derivatives are substituted for the modified derivatives. Hence, it is essentially Bonnet's condition.*

15. *Formulas of Transformation from one Orthogonal System to Another.* It will suffice for our purposes to develop the formulas of transformation from the lines of curvature C_1, C_2 to an arbitrary orthogonal system of curves C, C'.

Let the angle, at an arbitrary point P, from the directed curve C_1 to the directed curve C, measured in the positive direction, be $\alpha(u, v)$. Then the vectors ξ, ξ' may be obtained from the vectors $\xi^{(1)}$, $\xi^{(2)}$ by rotating these vectors about P in the tangent plane through the angle α. Thus:

$$\begin{aligned} \xi &= \xi^{(1)} \cos\alpha + \xi^{(2)} \sin\alpha, \\ \xi' &= -\xi^{(1)} \sin\alpha + \xi^{(2)} \cos\alpha. \end{aligned} \tag{48}$$

Since, by (29),

$$\begin{aligned} A'\xi + A\xi' &= A_2\xi^{(1)} + A_1\xi^{(2)}, \\ B'\xi + B\xi' &= B_2\xi^{(1)} + B_1\xi^{(2)}, \end{aligned}$$

we obtain, as the transformation of A_1du+B_1dv, A_2du+B_2dv into $Adu+Bdv$, $A'du+B'dv$:

$$\begin{aligned} Adu + Bdv &= (A_1du + B_1dv)\cos\alpha - (A_2du + B_2dv)\sin\alpha, \\ A'du + B'dv &= (A_1du + B_1dv)\sin\alpha + (A_2du + B_2dv)\cos\alpha. \end{aligned} \tag{49}$$

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* Condition (47) becomes, when for $1/\rho$ and $1/\rho'$ are substituted the value given by (16a),

$$\frac{\nabla}{\nabla s}\frac{\partial}{\partial s'}\log(\Delta_1\psi)^{1/2} - \frac{\nabla}{\nabla s'}\frac{\partial}{\partial s}\log(\Delta_1\phi)^{1/2} = 0,$$

or

$$\frac{\nabla}{\nabla s'}\frac{\partial}{\partial s}\log\left(\frac{\Delta_1\psi}{\Delta_1\phi}\right)^{1/2} = 0.$$

This is the invariant form of the usual condition to the effect that, when the linear element is Edu^2+Gdv^2, the parametric curves form an isometric system if and only if $\log(E/G)$ is of the form $U(u)+V(v)$.

The formulas of transformation of the ordinary directional derivatives are

$$\begin{aligned} \frac{\partial \chi}{\partial s} &= \frac{\partial \chi}{\partial s_1}\cos\alpha + \frac{\partial \chi}{\partial s_2}\sin\alpha, \\ \frac{\partial \chi}{\partial s'} &= -\frac{\partial \chi}{\partial s_1}\sin\alpha + \frac{\partial \chi}{\partial s_2}\cos\alpha, \end{aligned} \tag{50}$$

and those of the geodesic curvatures,

$$\begin{aligned} \frac{1}{\rho} - \frac{\partial \alpha}{\partial s} &= \frac{\cos\alpha}{\rho_1} + \frac{\sin\alpha}{\rho_2}, \\ \frac{1}{\rho'} - \frac{\partial \alpha}{\partial s'} &= -\frac{\sin\alpha}{\rho_1} + \frac{\cos\alpha}{\rho_2}. \end{aligned} \tag{51}$$

These formulas are readily obtained from (30) by replacing A, B, A', B' by their values in terms of A_1, B_1, A_2, B_2, as read off from (49).

The formulas for the transformation of the modified derivatives are now readily found to be

$$\begin{aligned} \frac{\nabla \chi}{\nabla s} - \chi\frac{\partial \alpha}{\partial s'} &= \frac{\nabla \chi}{\nabla s_1}\cos\alpha + \frac{\nabla \chi}{\nabla s_2}\sin\alpha, \\ \frac{\nabla \chi}{\nabla s'} + \chi\frac{\partial \alpha}{\partial s} &= -\frac{\nabla \chi}{\nabla s_1}\sin\alpha + \frac{\nabla \chi}{\nabla s_2}\cos\alpha. \end{aligned} \tag{52}$$

It is interesting, also, to note that Euler's formulas,

$$\frac{1}{r} = \frac{\cos^2\alpha}{r_1} + \frac{\sin^2\alpha}{r_2}, \qquad \frac{1}{r'} = \frac{\sin^2\alpha}{r_1} + \frac{\cos^2\alpha}{r_2}, \tag{53}$$

and the corresponding formula for the geodesic torsion, namely,

$$\frac{1}{\tau} = \frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sin 2\alpha, \tag{54}$$

are a direct result of our theory.

For example, if we set for ξ' and $\partial\zeta/\partial s$, in

$$\frac{1}{\tau} = \left(\xi' \,\middle|\, \frac{\partial \zeta}{\partial s}\right),$$

their values, as given by (48) and (50), in terms of $\xi^{(1)}$, $\xi^{(2)}$, $\partial\zeta/\partial s_1$, $\partial\zeta/\partial s_2$, and replace $\partial\zeta/\partial s_1$ and $\partial\zeta/\partial s_2$ by their values from (43), we obtain immediately formula (54).*

From (53) and (54) we verify that

$$K = \frac{1}{rr'} - \frac{1}{\tau^2}.$$

From these same formulas we also find the relation

$$\left(\frac{1}{r} - \frac{1}{r'}\right)\tan 2\alpha = \frac{2}{\tau},$$

which defines the angle α in terms of quantities pertaining only to the curves C and C'. Thus, when an orthogonal system of curves C and C' is given, as in §10, by its fundamental quantities, it is possible to determine precisely the position of the system with reference to the lines of curvature.

It is evident, conversely, that the foregoing formulas permit us to pass from the fundamental quantities A_1, B_1, A_2, B_2, $1/r_1$, $1/r_2$ for the lines of curvature to the fundamental quantities A, B, A', B', $1/r$, $1/r'$, $1/\tau$ for any orthogonal system of curves C and C', provided we know the angle $\alpha(u, v)$ under which the curves of this system meet the lines of curvature. In particular, the canonical differential equations of the curves C and C' are readily found, when those of the lines of curvature are known. But the canonical differential equations of the lines of curvature are easily obtained, by algebraic processes. In fact, we have

$$A_1^2 = \frac{1}{L}\left(\frac{E}{r_1} - e\right) \geqq 0, \qquad A_2^2 = \frac{1}{L}\left(e - \frac{E}{m}\right) \geqq 0,$$

$$A_1B_1 = \frac{1}{L}\left(\frac{F}{r_1} - f\right), \qquad A_2B_2 = \frac{1}{L}\left(f - \frac{F}{r_2}\right),$$

$$B_1^2 = \frac{1}{L}\left(\frac{G}{r_1} - g\right) \geqq 0, \qquad B_2^2 = \frac{1}{L}\left(g - \frac{G}{r_2}\right) \geqq 0,$$

where

* The general method of using directional derivatives to establish formulas (53) and (54) is ascribed to Enneper; see Lilienthal, Encyclopädie der Mathematischen Wissenschaften, III D3, p. 168.

$$L = \frac{1}{r_1} - \frac{1}{r_2}. \text{ *}$$

Since $A_2B_1 - A_1B_2 = D$, we may, in extracting the square roots of $A_1^2, B_1^2, A_2^2, B_2^2$, choose only one sign at pleasure. † The other three are then uniquely determined.

16. *Differential Parameters. The Significance of the Modified Directional Derivatives.* Let us consider, first, differential parameters with respect to the canonical linear forms $Adu+Bdv$ and $A'du+B'dv$, which, when equated to zero, define an orthogonal system of curves on S.

We single out two differential parameters of the first order with respect to these forms, namely, the derivatives of a function $\phi(u, v)$ in the positive directions of the curves C and C':

$$\frac{\partial \phi}{\partial s} = \frac{1}{D}(B\phi_u - A\phi_v), \qquad \frac{\partial \phi}{\partial s'} = -\frac{1}{D}(B'\phi_u - A'\phi_v),$$

and four differential parameters of the second order:

$$\frac{\nabla}{\nabla s}\frac{\partial \phi}{\partial s}, \quad \frac{\nabla}{\nabla s'}\frac{\partial \phi}{\partial s}, \quad \frac{\nabla}{\nabla s}\frac{\partial \phi}{\partial s'}, \quad \frac{\nabla}{\nabla s'}\frac{\partial \phi}{\partial s'},$$

the second and third of which are identical.

The fundamental differential parameters of the first and second order with respect to the linear element of the surface may be expressed in terms of these parameters related to the linear forms. We have

$$\Delta_1\phi = \left(\frac{\partial \phi}{\partial s}\right)^2 + \left(\frac{\partial \phi}{\partial s'}\right)^2,$$

$$\Delta_1(\phi,\psi) = \frac{\partial \phi}{\partial s}\frac{\partial \psi}{\partial s} + \frac{\partial \phi}{\partial s'}\frac{\partial \psi}{\partial s'}, \tag{55}$$

$$\Theta_1(\phi,\psi) = \frac{\partial \phi}{\partial s}\frac{\partial \psi}{\partial s'} - \frac{\partial \phi}{\partial s'}\frac{\partial \psi}{\partial s},$$

and

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* The cases of the sphere and the plane are excluded.

† Geometrically: we may fix the positive direction on the lines of curvatures of one family; then that for the other family is determined.

$$\Delta_2\phi = \frac{\nabla}{\nabla s}\frac{\partial\phi}{\partial s} + \frac{\nabla}{\nabla s'}\frac{\partial\phi}{\partial s'}. \tag{56}$$

To verify these equations, we have merely to expand the right-hand sides, and recall that

$$E = A^2 + A'^2, \quad F = AB + A'B', \quad G = B^2 + B'^2.$$

We are now in a position to develop on S the theory which is linked with the ideas of gradient, divergence, and curl, and to express it in an analytic form which is strikingly similar to that in the Cartesian plane. Let V be a vector which is defined at each point of S, and let the components of V with respect to ξ and ξ' be $Q(u, v)$ and $Q'(u, v)$,

$$V = Q\xi + Q'\xi',$$

The divergence and curl of the vector V shall be defined as the expressions:

$$\begin{aligned} \operatorname{div} V &= \frac{\nabla Q}{\nabla s} + \frac{\nabla Q'}{\nabla s'}, \\ \operatorname{curl} V &= \frac{\nabla Q'}{\nabla s} - \frac{\nabla Q}{\nabla s}. \end{aligned} \tag{57}$$

In justification of the definitions, we shall show that the expressions in question are invariant with respect to a transformation from one orthogonal system to a second. Let the components of V with respect to the unit vectors $\xi^{(1)}, \xi^{(2)}$ in the directions of the lines of curvature be Q_1 and Q_2:

$$V = Q_1\xi^{(1)} + Q_2\xi^{(2)}.$$

Using the formulas of transformation (48) of $\xi^{(1)}$, $\xi^{(2)}$ into ξ, ξ', we find

$$Q = Q_1 \cos\alpha + Q_2 \sin\alpha, \qquad Q' = -Q_1 \sin\alpha + Q_2 \cos\alpha,$$

and hence deduce, by means of the transformation (52) of the modified directional derivatives, the desired relations:

$$\begin{aligned} \frac{\nabla Q}{\nabla s} + \frac{\nabla Q'}{\nabla s'} &= \frac{\nabla Q_1}{\nabla s_1} + \frac{\nabla Q_2}{\nabla s_2}, \\ \frac{\nabla Q'}{\nabla s} - \frac{\nabla Q}{\nabla s'} &= \frac{\nabla Q_2}{\nabla s_1} - \frac{\nabla Q_1}{\nabla s_2}. \end{aligned}$$

The expressions (57) are the inner and outer products* of the vector operator

$$\Delta = \xi \frac{\nabla}{\nabla s} + \xi' \frac{\Delta}{\nabla s'}$$

with the vector V. This operator is, then, a generalization of the vector operator in the plane which is denoted by the same symbol.

Another vector operator suggests itself, namely,

$$D = \xi \frac{\partial}{\partial s} + \xi' \frac{\partial}{\partial s'},$$

and this operator, too, is of value. In fact, if $\phi(u, v)$ is a scalar function, the vector

$$D\phi = \frac{\partial \phi}{\partial s} \xi + \frac{\partial \phi}{\partial s'} \xi'$$

is the gradient of ϕ.

The operator D, based on ordinary directional derivatives, applies to functions and yields vectors,—their gradients. The operator ∇, based on the modified directional derivatives, applies to vectors and yields functions,—their curls and divergence.

This result sets forth clearly the significance of the modified derivatives and the relationship between them and the ordinary derivatives.

Since D applies to functions and yields vectors, and ∇ applies to vectors and yields functions, the two operators can be applied *in turn* as often as one desires.

We can, for example, apply ∇ to the gradient $D\phi$ of a function ϕ, to find its divergence and curl.

The divergence of the gradient of $\phi(u, v)$ is the Laplacian

$$\operatorname{div} D\phi \equiv \frac{\nabla}{\nabla s} \frac{\partial \phi}{\partial s} + \frac{\nabla}{\nabla s'} \frac{\partial \phi}{\partial s'}.$$

The curl of the gradient of $\phi(u, v)$ is identically zero,

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* We assume hat we are dealing with two-dimensional vectors. Then the outer product, as well as the inner product, is a scalar.

$$\text{curl } D\phi \equiv \frac{\nabla}{\nabla s}\frac{\partial \phi}{\partial s'} - \frac{\nabla}{\nabla s'}\frac{\partial \phi}{\partial s} \equiv 0.$$

Conversely, *if the curl of a vector vanishes identically, the vector is the gradient of a function.*

These results are, of course, direct generalizations of classical theorems in the Cartesian plane.

Since the gradients $D\phi$, $D\psi$ of two invariant functions are invariant vectors, their inner and outer products are invariant functions. In fact, according to (55), these products are the differential parameters of the first order of ϕ, ψ with respect to the linear element. On the other hand, the divergence of $D\phi$ is $\Delta_2\phi$.

All the differential parameters, with respect to the linear element, of a given set of functions result from the operations of inner and outer multiplication, together with the operation of finding divergence, applied to the gradients of the given functions and all functions derived from them by these operations.

The process of finding the divergence is simply that of inner multiplication by the operator ∇. Hence, we have reduced the operations necessary to obtain the differential parameters with respect to the linear element to ordinary inner and outer multiplication of two-dimensional vectors, supplemented by the two types of differentiation.

17. *The Geodesic Curvature Vector.* The vector whose components in the directions of the curves of an orthogonal system to which the surface is referred are the geodesic curvatures of these curves shall be called the geodesic curvature vector associated with the orthogonal system. Thus,

$$\Gamma_{C_1C_2} = \frac{1}{\rho_1}\xi^{(1)} + \frac{1}{\rho_2}\xi^{(2)}$$

is the geodesic curvature vector associated with the lines of curvature, and

$$\Gamma_{C_1C'} = \frac{1}{\rho}\xi + \frac{1}{\rho'}\xi'$$

is that associated with the general orthogonal system consisting of the curves C, C'.

These two geodesic curvature vectors are not, in general, the same. As a matter of fact, we have

$$\Gamma_{CC'} = \Gamma_{C_1C_2} + D\alpha,$$

where $D\alpha$ is the gradient of the angle α under which the system of curves C, C' cuts the lines of curvature.

On the other hand, since the curl of the gradient of a function vanishes,

$$\text{curl } \Gamma_{CC'} = \text{curl } \Gamma_{C_1C_2}.$$

The common value of these curls is, by (40), the negative of the total curvature of the surface.

The curl of the geodesic curvature vector is the same for every orthogonal system of curves, and is equal to the negative of the total curvature of the surface.

It follows that a surface is a developable if and only if an arbitrarily chosen geodesic curvature vector is the gradient of a function.

HARVARD UNIVERSITY

THE RATIONALITY OF CERTAIN CONTINUOUS CURVES

BY G. T. WHYBURN

1. *Introduction.* In this paper we establish the following theorem.

THEOREM. *Every plane continuous curve M every subcontinuum of which is a continuous curve is a rational curve.*†

We first state two lemmas, both of which are very easy to prove.

LEMMA 1. *In order that any continuous curve M should be a rational curve it is necessary and sufficient that every maximal cyclic curve of M be a rational curve.*

LEMMA 2. *If P is any point of a plane continuous curve M, every subcontinuum of which is a continuous curve, then there exists a continuous curve M^*, also every subcontinuum of which is a continuous curve, which contains M and such that P is on the boundary of no complementary domain of M^*.*

The necessity of the condition in Lemma 1 is obvious. The sufficiency is established by the following steps: (a) a continuum M is a rational curve if and only if every two points A and B of M can be separated in M by a countable subset of M. (Menger, loc. cit., proves this where A and B are closed sets; a simple application of the Lindelöf Theorem suffices to prove the part not included in Menger's theorem); (b) if two points A and B of a continuous curve M lying together in a maximal cyclic curve C of M are separated in C by a closed subset K of C, then K also separates A and B in M, (because if not, then A and B lie together in a component H of $M-K$, and since $H \cdot C$

† A continuum M is said to be a rational curve if each point of M is contained in arbitrarily small neighborhoods in M with countable boundaries (see K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), p. 277); or, in other words, for each point P of M and each $\epsilon>0$, P can be ϵ-separated in M by a countable subset of M (see P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, II Verhandelingen der Akademie Amsterdam, vol. 13, No. 4).

is connected and contains $A+B$, this contradicts the assumption that K separates A and B in C); (c) if two points A and B of a continuous curve M do not lie together in a maximal cyclic curve of M, there exists at least *one point* of M which separates A and B in M. Under our hypothesis it follows by (b) and (c) that every two points of M can be separated in M by a countable subset of M; and hence, by (a), M is a rational curve. Lemma 2 is readily established by constructing, in each complementary domain D of M having the given point P on its boundary, a sequence of arc-crosscuts converging to P such that P is a limit point of no single component of D minus the sum of these crosscuts, and these are added to M to form M^*.

2. *Proof of the Theorem.* Let P be any point of M and ϵ any positive number. By Lemma 1 we may assume M to be cyclicly connected. And by Lemma 2, we may assume, also without loss of generality, that P is on the boundary of no complementary domain of M; for if this is not the case, we consider the curve M^* and show that M^* is rational at P; and then since $M^* \supset M$, it follows that M also is rational at P. By a theorem of the author's† there exists a simple closed curve J of diameter $<\epsilon/4$ enclosing P and such that if K denotes the sum of the boundaries of all those complementary domains of M each of which has a boundary point on J, then $J \subset K$ and $\delta(K)<\epsilon/4$. Since K is a subcontinuum of M, it must be a continuous curve. Let C be the maximal cyclic curve of K containing J, and let B be the boundary of the complementary domain of C which contains P. Let $[I_i]$, $i=1, 2, 3, \cdots$ be the collection of complementary domains of M such that if, for each i, J_i denotes the boundary of I_i then J_i belongs to K and contains at least one point of B. Then since $B \subset K$ and every point of K is a boundary point of at least one complementary domain of M, therefore $B \subset \sum_1^\infty J_i$. Since C and M are cyclicly connected, therefore‡ each of the curves $B, J_1, J_2, J_3, \cdots$ is a simple closed curve. Now since $B \subset M$, it follows that, for each i, the interior I_i of J_i

† *Concerning the complementary domains of continua*, Annals of Mathematics, vol. 29 (1928), pp. 399–411, Theorem 18.

‡ G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38, Theorem 10.

lies either wholly within or wholly without B; and if, for any i, I_i lies within B, then $B \cdot J_i$ consists of exactly one point. Thus if we divide the curves of the collection $[J_i]$ into two classes G_1 and G_2 such that, for each i, J_i belongs to G_1 or to G_2 according as $B \cdot J_i$ consists of a single point or of more than one point, then the interior of every curve of G_2 lies wholly without B.

Now let Q be a point at a distance $\geqq \epsilon$ from P. Then Q is without every one of the curves $B, J_1, J_2, \cdots$. Let $J_{n_1}, J_{n_2}, \cdots$ be those curves of class G_2 each of which contains a point A_{n_i} which is accessible from Q by an arc having only A_{n_i} in common with $\sum J_i$. For each i, let X_i and Y_i be the boundary points of the component of $J_{n_i} - J_{n_i} \cdot B$ which contains A_{n_i}. For each i, $I_{n_i} + X_i + Y_i$ contains an arc T_i from X_i to Y_i. Let H denote the closed point set $\overline{\sum T_i}$. Then clearly $H \subset \sum T_i + B$. Now if E is any point of H not in $\sum T_i$, it is easily seen that E must be accessible from Q by an arc having only E in common with $\sum J_i$. And since E belongs to no arc T_i, clearly it can then belong to no curve of class G_2. Hence it must belong to a curve of class G_1; and thus there exists an integer j such that $E = B \cdot J_j$. Therefore the set of all such points $[E]$ is countable; and since, for each i, $T_i \cdot M = X_i + Y_i$, it follows that $H \cdot M$ is countable.

Now let Z be any point at a distance $\geqq \epsilon$ from P. Then Z and Q lie in the same complementary domain of H, since H lies in a circle with center P and radius $\epsilon/2$. However, the set H must separate P and Z. For if not, there exists an arc ZP from Z to P with $H \cdot ZP = Z$. Let V be the first point on ZP, in the order from Z to P, which belongs to B. Since $ZV - V$ contains no point of any of the arcs T_i, it follows that V is accessible from Q and belongs to no curve of class G_2. But since the set of all such points of B is countable, it is readily seen that every such point is a limit point of $\sum T_i$ and hence belongs to H, contrary to supposition. Thus H ϵ-separates P in the plane, and since $H \cdot M$ is countable it follows that M is rational at P, and thus our theorem is established.

In conclusion I will remark that the above proof depends very sharply upon properties of the plane, and I do not know whether or not the theorem is true in a space of more dimensions, but would be very much interested in a solution for such a space.

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NOTE ON RULED SURFACES AND THEIR DEVELOPABLES*

BY A. F. CARPENTER

If a plane π, whose coordinates κ, λ, μ, ν, are functions of the independent variable x in the system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

defining a ruled surface R, is to be fixed relative to R, then must these coordinates satisfy the relations

$$(2)\qquad \begin{aligned} 2\kappa' &= \eta\kappa - p_{12}\lambda + \mu, \\ 2\lambda' &= -p_{21}\kappa + \eta\lambda + \nu, \\ 2\mu' &= (p_{12}p_{21} - 4q_{11})\kappa + \eta\mu - p_{12}\nu, \\ 2\nu' &= (p_{12}p_{21} - 4q_{22})\lambda - p_{21}\mu + \eta\nu, \end{aligned}$$

where η is an arbitrary function of x.†

The pole of this plane with respect to the quadric Q,

$$(3)\qquad x_1x_4 - x_2x_3 = 0,$$

which osculates R along a line element l_{yz} is given by the expression

$$(4)\qquad \theta = \nu y - \mu z - \lambda\rho + \kappa\sigma$$

where

$$(5)\qquad \rho = 2y' + p_{12}z, \qquad \sigma = 2z' + p_{21}y,$$

and the point which corresponds to π in the null-system of the linear complex which osculates R along l_{yz} is given by the expression

$$(6)\qquad \phi = -p_{21}\mu y + p_{12}\nu z + p_{21}\kappa\rho - p_{12}\lambda\sigma.$$

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* Presented to the Society, June 20, 1930.

† Carpenter, *Some fundamental relations in the projective differential geometry of ruled surfaces*, Annali di Matematica, (3), vol. 26.

The equations of the quadrics Q_1, Q_2, which osculate the two branches of R's flecnode surface along those line elements which intersect l_{yz}, are

$$p_{12}(x_1x_4 - x_2x_3) - 2qx_4^2 = 0, \tag{7}$$

$$p_{21}(x_1x_4 - x_2x_3) - 2qx_3^2 = 0, \tag{8}$$

where $q = q_{11} - q_{22}$.

The poles of π with respect to these two quadrics are given respectively by the expressions

$$\theta_1 = (4q\kappa + p_{12}\nu)y - p_{12}\mu z - p_{12}\lambda\rho + p_{12}\kappa\sigma = 4q\kappa y + p_{12}\theta, \tag{9}$$

$$\theta_2 = p_{21}\nu y + (4q\lambda - p_{21}\mu)z - p_{21}\lambda\rho + p_{21}\kappa\sigma = 4q\lambda z + p_{21}\theta. \tag{10}$$

From (9) and (10) it results that the lines $l_{\theta\theta_1}$, $l_{\theta\theta_2}$ pass through the respective points y, z and hence that the plane π, determined by these three points, contains l_{yz}. Its equation is found to be

$$\kappa x_3 + \lambda x_4 = 0. \tag{11}$$

The one-parameter family of planes π_1, one for each line element of R, determines a developable surface. The generators of this surface are the lines $l_{\alpha\theta}$ where

$$\alpha = \lambda y - \kappa z \tag{12}$$

is the point of intersection of plane π with l_{yz}. This we show by finding the characteristic line of plane π_1.

By making use of conditions (2) the equation of the plane determined by the points $\theta+d\theta$, $\theta_1+d\theta_1$, $\theta_2+d\theta_2$ is found to be

$$q[\kappa^2\lambda x_1 + \kappa\lambda^2 x_2 - (2\kappa\lambda\mu + \kappa^2\nu)x_3 - (2\kappa\lambda\nu + \lambda^2\mu)x_4]dx \\ - [2q\kappa\lambda + (3q\kappa\lambda\eta + 4q'\kappa\lambda - p_{12}q\lambda^2 - p_{21}q\kappa^2)dx](\kappa x_3 + \lambda x_4) = 0. \tag{13}$$

From (11) and (13) we find that this characteristic line is determined by the pair of planes

$$\kappa^2\lambda x_1 + \kappa\lambda^2 x_2 - \kappa(2\lambda\mu + \kappa\nu)x_3 - \lambda(2k\nu + \lambda\mu)x_4 = 0,$$
$$\kappa x_3 + \lambda x_4 = 0.$$

The points α, θ are seen to lie in both planes.

Again, by making use of (2), we find

(14) $2pA\theta' + 2qB\phi' - p(q\kappa\lambda + A\eta)\theta - q(B\eta + 4C)\phi = 0,$

(15) $2A\theta' - 4q\kappa\lambda\alpha' - (A\eta + 2q\kappa\lambda)\theta + 2q(\kappa\lambda\eta + \kappa\nu + \lambda\mu)\alpha = 0,$

where

$$p = p_{12}q_{21} - p_{21}q_{12}, \quad A = \kappa\nu - \lambda\mu, \quad B = p_{21}\kappa^2 - p_{12}\lambda^2,$$
$$C = q_{21}\kappa^2 - q_{12}\lambda^2.$$

Equation (14) expresses the condition that the line $l_{\theta\phi}$ shall generate a developable. Equation (15) is the similar condition for $l_{\alpha\theta}$.

The focal points of these respective lines are seen to be

(16) $$\beta = pA\theta + qB\phi, \quad \gamma = A\theta - 2q\kappa\lambda\alpha.$$

Combining the above results we may state the following theorem.

THEOREM 1. *Each plane π fixed in position with respect to a ruled surface R determines with R two developable surfaces. One cuts π in the curve of intersection of π and R, the other cuts π in a curve whose points are those which correspond to π in the null-systems of the osculating linear complexes of R, and they intersect each other in a curve whose points are the poles of π with respect to the quadrics which osculate R along its line elements.*

The plane π_1 is tangent to Q at the point α, and its null-point as determined by the linear complex osculating R along lyz is given by the expression

(17) $$\kappa p_{21}y - \lambda p_{12}z.$$

This point is on the line l_{yz} and is in fact the point of intersection of l_{yz} and $l_{\theta_1\theta_2}$, since

$$4q(\kappa p_{21}y - \lambda p_{12}z) = p_{21}\theta_1 - p_{12}\theta_2.$$

The equation of the plane of the points θ, ϕ, θ_1 is

$$Bx_2 - p_{12}Ax_3 + Dx_4 = 0,$$

where $D = p_{21}\kappa\mu - p_{12}\lambda\nu$. Its pole with respect to Q is given by

(18) $$Dy + p_{12}Az - B\rho$$

and its null-point by

(19) $$p_{21}Ay + Dz - B\sigma.$$

The equation of the plane of the points θ, ϕ, θ_2 is

$$Bx_1 + Dx_3 - p_{21}Ax_4 = 0.$$

Its pole with respect to Q is given by

$$p_{21}Ay + Dz - B\sigma \tag{20}$$

and its null-point by

$$Dy + p_{12}Az - B\rho. \tag{21}$$

In view of (18), (19), (20), (21) we may state the following theorem.

THEOREM 2. *The polar reciprocal of the line $l_{\theta\phi}$ with respect to Q is identical with its polar reciprocal as determined by the linear complex osculating R along l_{yz}.*

The equation of the plane of the points ϕ, θ_1, θ_2, is

$$B(p_{12}\lambda x_1 + p_{21}\kappa x_2) - p_{12}[(p_{21}\kappa^2 + p_{12}\lambda^2)\nu - 2p_{21}\kappa\lambda\mu + 4q\kappa\lambda^2]x_3 + p_{21}[(p_{21}\kappa^2 + p_{12}\lambda^2)\mu - 2p_{12}\kappa\lambda\nu - 4q\kappa^2\lambda]x_4 = 0,$$

the null-point of this plane is given by

$$[(p_{21}\kappa^2 + p_{12}\lambda^2)\nu - 2\rho_{21}\kappa\lambda\mu + 4q\kappa\lambda^2]y + [(p_{21}\kappa^2 + p_{12}\lambda^2)\mu - 2p_{12}\kappa\lambda\nu - 4q\kappa^2\lambda]z + B(\lambda\rho - \kappa\sigma),$$

and this expression, when multiplied by $2q$, becomes

$$(2q\lambda^2 + p_{21}A)\theta_1 - (2q\kappa^2 + p_{12}A)\theta_2.$$

Moreover the coordinates of this null-point satisfy the equation

$$\kappa x_1 + \lambda x_2 + \mu x_3 + \nu x_4 = 0,$$

of the plane π. We have thus the following theorem.

THEOREM 3. *The null-point of the plane determined by the null-point of π and π's two poles with respect Q_1 and Q_2, lies on the line $l_{\theta_1\theta_2}$ at the point where it cuts π.*

Many other interesting properties of this tetrahedron θ, ϕ, θ_1, θ_2, determined by π and R, can be obtained by methods similar to the above.

THE UNIVERSITY OF WASHINGTON

ON UNIFORM SUMMABILITY OF SEQUENCES OF CONTINUOUS FUNCTIONS*

BY R. P. AGNEW

In a paper by D. C. Gillespie and W. A. Hurwitz,† it was shown that any bounded real sequence $\{s_n(x)\}$ of continuous functions, defined over a closed compact set A in a metric space, which converges over A to a continuous function $s(x)$, is uniformly summable to $s(x)$ by a regular transformation. The transformations used were transformations with square matrices which were constructed in terms of the elements of the $\{s_n(x)\}$ sequences considered; hence different transformations were used to effect the uniform summability of different sequences. In §8 the question was raised as to whether some restricted class of regular transformations of a familiar type might serve for the uniform summability of all sequences of the class considered; the question was partially answered by exhibiting a single sequence for which the Cesàro, Euler-Abel, and Borel transformations will not suffice. It is the object of this note to show that also no one regular transformation whatever of the general form considered below will suffice.

Nearly every transformation that has been considered in the theory of summability‡ is a special case of the following one. Let T and A be sets in metric spaces, let T have a limit point t_0 not belonging to T, and let the functions $a_k(t), k=1, 2, 3, \cdots$ be defined over T. Then if a sequence $\{s_n(x)\}$, defined over A, is such that

$$(G) \qquad \sigma(t, x) = \sum_{k=1}^{\infty} a_k(t) s_k(x)$$

converges for all t in T and x in A and if

* Presented to the Society, April 18, 1930.

† Transactions of the American Mathematical Society, vol. 32 (1930), pp. 527–543.

‡ For a bibliography of the subject, see L. L. Smail, *History and Synopsis of the Theory of Summable Infinite Processes*, University of Oregon Press, 1925.

$$\lim_{t \to t_0(T)} \sigma(t, x) = \sigma(x), \tag{1}$$

the sequence $\{s_n(x)\}$ is said to be summable (G) to $\sigma(x)$; if the limit is approached uniformly with respect to x in (1), then $\{s_n(x)\}$ is said to be uniformly summable (G) to $\sigma(x)$.

The answer to the question we are considering is contained in the following theorem.

THEOREM 1. *In order that (T) may be such that*

$$\lim_{t \to t_0(T)} \sigma(t, x) = 0, \quad \textit{uniformly with respect to } x \tag{2}$$

*for every sequence $\{s_n(x)\}$ of continuous functions, defined over an infinite set A, such that $s_n(x)$ is bounded over A for all n and $\lim_{n \to \infty} s_n(x) = 0$ over A, it is necessary and sufficient that**

$$\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} | a_k(t) | = 0. \tag{3}$$

To establish necessity, let (G) be a given transformation which implies (2) for every admissible sequence $\{s_n(x)\}$. Considering, for each positive integer k, the sequence $s_n(x) = 0$, $n \neq k$, and $s_k(x) = 1$, we see that (G) must satisfy the condition

$$\text{for each } k, \quad \lim_{t \to t_0(T)} a_k(t) = 0. \tag{4}$$

We shall show that (3) is a necessary condition by supposing that (G) satisfies (4) but not (3) and defining an admissible sequence $\{s_n(x)\}$ for which (2) fails. Since (3) is denied there is a number $\theta > 0$ such that

$$\limsup_{t \to t_0(T)} \sum_{k=1}^{\infty} | a_k(t) | > \theta;$$

hence there is a sequence $\{t_n\}$ of points of T with the limit t_0 such that

$$\limsup_{n \to \infty} \sum_{k=1}^{\infty} | a_k(t_n) | > \theta. \tag{5}$$

Using (5), choose an index n_1 such that

* A transformation (G) which satisfies (3) is not regular; it is a null transformation.

$$\sum_{k=1}^{\infty} | a_k(t_{n_1}) | > \theta$$

and choose $N_1 > n_1$ such that

$$\sum_{k=1}^{N_1} | a_k(t_{n_1}) | > \theta.$$

Using (4) and (5), choose $n_2 > N_1$ such that

$$\sum_{k=1}^{N_1} | a_k(t_\alpha) | < \theta/2 \text{ for } \alpha \geqq n_2 \text{ and } \sum_{k=1}^{\infty} | a_k(t_{n_2}) | > \theta,$$

and choose $N_2 > n_2$ such that

$$\sum_{k=1}^{N_2} | a_k(t_{n_2}) | > \theta.$$

Proceeding in this manner, choose $n_1 < N_1 < n_2 < N_2 < n_3 < \cdots$ such that for $p = 2, 3, 4, \cdots$

$$\sum_{k=1}^{N_{p-1}} | a_k(t_\alpha) | < \theta/2 \text{ for } \alpha \geqq n_p \text{ and } \sum_{k=1}^{N_p} | a_k(t_{n_p}) | > \theta.$$

Then, defining $N_0 = 0$, we have

$$\sum_{k=N_{p-1}+1}^{N_p} | a_k(t_{n_p}) | > \frac{\theta}{2}, \quad p = 1, 2, 3, \cdots. \tag{6}$$

Let $\{x_p\}$ be a sequence of distinct points of A such that no point of $\{x_p\}$ is a limit point of $\{x_p\}$. Then corresponding to each point x_p of $\{x_p\}$ there is a positive number r_p such that $r(x_p, x_q) > 2r_p$* for $q \neq p$; let the set of points x of A for which $r(x_p, x) < r_p$ be denoted by A_p. Then $A_1, A_2, A_3, \cdots$ are mutually exclusive subsets of A. Define the sequence $\{s_p(x)\}$ over A as follows: For k such that $N_{p-1} < k \leqq N_p$ and for $p = 1, 2, 3, \cdots$

$$s_k(x) = \begin{cases} [1 - r(x_p, x)/r_p] \operatorname{sgn} a_k(t_{n_p}) & \text{over } A_p, \\ 0 & \text{over } A - A_p. \end{cases}$$

* $r(a_1, a_2)$ is used to denote the distance between two points a_1 and a_2 of A.

It is readily seen that $s_n(x)$ is continuous over A for each n, $s_n(x)$ is bounded over A for all n and $\lim s_n(x)=0$ over A; hence $\{s_n(x)\}$ is an admissible sequence. But for each index p

$$s_k(x_p) = \begin{cases} \operatorname{sgn} a_k(t_{n_p}) \text{ for } N_{p-1} < k \leqq N_p, \\ 0 \quad \text{for} \quad k \leqq N_{p-1} \quad \text{and} \quad k > N_p, \end{cases}$$

and hence for each index p

$$\sigma(t_{n_p}, x_p) = \sum_{k=N_{p-1}+1}^{N_p} a_k(t_{n_p}) \operatorname{sgn} a_k(t_{n_p}) = \sum_{k=N_{p-1}+1}^{N_p} |a_k(t_{n_p})| .$$

Using (6) and the preceding relation we obtain

$$\limsup_{p\to\infty} \sigma(t_{n_p}, x_p) \geqq \theta/2;$$

thus (2) is denied and the necessity of (3) is established.

The sufficiency of (3) is readily established by proving the following theorem.

THEOREM 2. *In order that* (*G*) *may be such that* (2) *is implied for every bounded sequence of functions,* (3) *is sufficient.*

Let (*G*) satisfy (3) and let M be a constant such that $|s_n(x)| < M$ over A for all n. Then

$$|\sigma(t, x)| \leqq \sum_{k=1}^{\infty} |a_k(t)|\,|s_k(x)| \leqq M \sum_{k=1}^{\infty} |a_k(t)|$$

and (2) follows.

CORNELL UNIVERSITY

ON CERTAIN RULED HYPERSURFACES OF ORDER $k-1$ IN r-SPACE

BY B. C. WONG

This note proposes to prove the following theorem.

THEOREM 1. *The locus of the lines incident with k given linear mutually skew sub-spaces of dimensions $t_1, t_2, \cdots, t_k$, respectively, in r-space such that*

$$k-2 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_k \leqq r-2, \tag{1}$$

and

$$t_1 + t_2 + \cdots + t_k = rk - r - k \tag{2}$$

is a hypersurface of order $k-1$.

As an immediate consequence of this theorem we have the following.

THEOREM 2. *The order of the ruled hypersurface whose generators have simple incidence with each of k varieties of orders $n_1, n_2, \cdots, n_k$ and dimensions $t_1, t_2, \cdots, t_k$ respectively in r-space where r, k, t_1, etc., satisfy (1) and (2) is $(k-1)n_1 n_2 \cdots n_k$.*

The case for which $k=r$ and $t_1=t_2=\cdots=t_k=r-2$ is known.* Thus, if $k=r=3$ and $t_1=t_2=t_3=1$, we have a ruled surface of order $2n_1n_2n_3$ whose rulings meet three curves of orders n_1, n_2, n_3 respectively in 3-space.† The theorems stated above include all the cases for which $2 \leqq k \leqq r$. Thus, if $k=4$, the locus of the lines incident with four sub-spaces of appropriate dimensions in r-space is always a cubic hypersurface. We need prove Theorem 1 only, since Theorem 2 follows from it immediately.

To avoid too complicated a system of notations we use $\{x\}$ in place of S_x to designate an x-space. Then the given k sub-

* Segre, Encycklopädie der Mathematischen Wissenschaften, vol. III C7, pp. 815, 832. Also B. C. Wong, *On a certain system of $r-2$ lines in r-space*, and *On the loci of the lines incident with k $(r-2)$-spaces in S_r*, this Bulletin, vol. 34 (1928), pp. 553–554, and pp. 715–717.

† Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. II, §467.

spaces of dimensions $t_1, t_2, \cdots, t_k$ will be denoted by $\{t_1\}$, $\{t_2\}, \cdots, \{t_k\}$ respectively.

Let N be the order of the hypersurface V_{r-1}^N in r-space satisfying the conditions imposed. We are to prove

$$N = k - 1.$$

A (t_1+1)-space $\{t_1+1\}$ passing through $\{t_1\}$ meets the other $k-1$ sub-spaces $\{t_i\}$ in the spaces $\{t_i'\}$ where $t_i' = t_1+t_i+1-r$ $[i=2, 3, \cdots, k]$. The lines which meet $\{t_i'\}$ form a $V_{t_1}^{N-1}$ in $\{t_1+1\}$ which together with $\{t_1\}$ constitutes the intersection of $\{t_1+1\}$ and V_{r-1}^N. Now a $(t_2'+1)$-space $\{t_2'+1\}$ $[t_2' = t_1+t_2+1-r]$ passing through $\{t_2'\}$ and belonging to $\{t_1+1\}$ meets the other $k-2$ t_i'-spaces of $\{t_1+1\}$ in the spaces $\{t_i''\}$ where $t_i'' = t_1+t_2+t_i+2-2r[i=3, 4, \cdots, k]$. This $\{t_2'+1\}$ meets $V_{t_1}^{N-1}$ in $\{t_2'\}$ and a $V_{t_2'}^{N-2}$ which is the locus of the lines incident with the spaces $\{t_i''\}$.

We have so far gone through two steps in the reasoning. Continuing this process, we soon arrive at the jth step that a $(t_j^{(j-1)}+1)$-space $\{t_j^{(j-1)}+1\}$ passing through $\{t_j^{(j-1)}\}$ where $t_j^{(j-1)} = t_1+t_2+\cdots+t_j+(j-1)(1-r)$ and belonging to $\{t_{j-1}^{(j-1)}\}$ meets the other $k-j$ $(t_i^{(j-1)})$-spaces of $\{t_{j-1}^{(j-2)}\}$ in the spaces $\{t_i^{(j)}\}$ where $t_i^{(j)} = t_1+t_2+\cdots+t_j+t_i+j(1-r)$ $[i=j+1, j+2, \cdots, k]$. The locus of the lines meeting $\{t_i^{(j)}\}$ is a V_m^{N-j} where $m = t_j^{(j-1)}$ in $\{m+1\} \equiv \{t_j^{(j-1)}+1\}$. This V_m^{N-j} and $\{m\} \equiv \{t_j^{(j-1)}\}$ form the complete intersection of $\{m+1\}$ with $V_{m'}^{N-j+1}$ where $m' = t_{j-1}^{(j-2)}$ which is supposed to have been obtained in the $(j-1)$th step.

Now let $j=k-2$. We have a $\{t_{k-2}^{(k-3)}+1\}$ or $\{T\}$ $[T = t_{k-2}^{(k-3)}+1 = 2r-2-t_k-t_{k-1}$ on account of (2)] passing through $\{T-1\}$ and belonging to $\{t_{k-3}^{(k-4)}\}$ which meets the other two spaces $\{t_{k-1}^{(k-3)}\}$ and $\{t_k^{(k-3)}\}$ of $\{t_{k-3}^{(k-4)}\}$ in the spaces $\{T'\}$ and $\{T''\}$ respectively where

$$T' = t_{k-1}^{(k-2)} = r-2-t_k$$

and

$$T'' = t_k^{(k-2)} = r-2-t_{k-1}.$$

The locus of lines incident with $\{T'\}$ and $\{T''\}$ is a V_{T-1}^{N-k+2} in $\{T\}$. Since the locus of lines meeting two sub-spaces $\{n_1\}$,

$\{n_2\}$ of $\{n\}$ such that $n=n_1+n_2+2$ is an $\{n-1\}$, the V_{T-1}^{N-k+2} is a $\{T-1\}$ in $\{T\}$, for $T=T'+T''+2$. Hence, we have

$$N - k + 2 = 1,$$

or

$$N = k - 1,$$

which was to be proved.

It is about impossible to state generally any proposition concerning the properties of the hypersurfaces of this type. The properties of some of them can be obtained readily.

THE UNIVERSITY OF CALIFORNIA

A NECESSARY AND SUFFICIENT CONDITION FOR THE NON-EQUIVALENCE OF ANY TWO RATIONAL GENERALIZED QUATERNION DIVISION ALGEBRAS*

BY A. A. ALBERT

Two algebras $\mathfrak{A}$ and $\mathfrak{B}$ over the same field F are called *equivalent* (or *simply isomorphic*) if it is possible to establish between their quantities a (1-1) correspondence such that if any quantities x and y of $\mathfrak{A}$ correspond to X and Y of $\mathfrak{B}$, then $x+y$, xy and αx correspond to $X+Y$, XY, αX respectively for every α of F. We shall consider two generalized quaternion division algebras† $\mathfrak{A}$ and $\mathfrak{B}$ over the field of all rational numbers, R. Let $\mathfrak{A}$ be given by

$$(1) \quad \mathfrak{A} = (e, i_1, j_1, i_1 j_1), \; i_1^2 = \rho_1 e, \; j_1^2 = \sigma_1 e, \; j_1 i_1 = -i_1 j_1,$$

where e is the modulus of $\mathfrak{A}$, and ρ_1 and σ_1 are in R. Without loss of generality ρ_1 and σ_1 may be taken to be each products of distinct rational prime integers.

Similarly let

$$(2) \quad \mathfrak{B} = (E, I_1, J_1, I_1 J_1), \; I_1^2 = \rho_2 E, \; J_1^2 = \sigma_2 E, \; J_1 I_1 = -I_1 J_1,$$

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* Presented to the Society, February 22, 1930.

† For the definition and properties of these algebras see L. E. Dickson, *Algebren und ihre Zahlentheorie*, pp. 46–49.

where ρ_2 and σ_2 are each products of distinct rational prime integers, and E is the modulus of $\mathfrak{B}$. Without loss of generality we may take $\sigma_2<0$ since one of $\rho_2,\sigma_2,-\rho_2\sigma_2$ is negative and by changing the roles of I_1, J_1, $I_1 J_1$ we may take σ_2 to be the negative number. The rank equation of $\mathfrak{A}$ is known (loc. cit.) to be

$$\omega^2 - 2\alpha_4\omega + (\alpha_4^2 - \alpha_1^2\rho_1 - \alpha_2^2\sigma_1 + \alpha_3^2\sigma_1\rho_1) = 0, \tag{3}$$

where the general quantity of $\mathfrak{A}$ is $x=\alpha_4+\alpha_1 i_1+\alpha_2 j_1+\alpha_3 i_1 j_1$. We have obviously the following property

LEMMA 1. *A quantity x which is not a rational multiple of e has a rational multiple of e as its square if and only if*

$$x = \alpha_1 i_1 + \alpha_2 j_1 + \alpha_3 i_1 j_1, \tag{4}$$

in which case

$$x^2 = (\alpha_1^2\rho_1 + \alpha_2^2\sigma_1 - \alpha_3^2\sigma_1\rho_1)e. \tag{5}$$

Consider the form

$$q = \alpha_1^2\rho_1 + \alpha_2^2\sigma_1 - \alpha_3^2\sigma_1\rho_1 - \alpha_4^2\rho_2 - \alpha_5^2\sigma_2. \tag{6}$$

The signs of ρ_1, σ_1 and $-\rho_1\sigma_1$ are the same if and only if $\rho_1<0$, $\sigma_1<0$. But we have taken $\sigma_2<0$, whence $-\sigma_2>0$ and has a sign different from that of $\rho_1<0$. The signs of the coefficients of q are therefore not all alike and they are all different from zero when $\mathfrak{A}$ and $\mathfrak{B}$ are division algebras. But every indefinite quadratic form in five variables is a null form* so that there exist integers $\alpha_1, \cdots, \alpha_5$, not all zero, for which $q=0$. Let $\alpha_{12}, \alpha_{22}, \cdots, \alpha_{52}$ be *any* set of integer solutions of $q=0$. The quantity $\rho'=\alpha_{42}^2\rho_2 + \alpha_{52}^2\sigma_2$ is a rational integer equal to $\alpha_{12}^2\rho_1+\alpha_{22}^2\sigma_1-\alpha_{43}^2\sigma_1\rho_1$.

The quantities

$$i' = \alpha_{12} i_1 + \alpha_{22} j_1 + \alpha_{32} i_1 j_1, \quad I' = \alpha_{42} I_1 + \alpha_{52} J_1$$

have the properties that

$$(i')^2 = \rho' e, \qquad (I')^2 = \rho' E.$$

Since $\alpha_{12}, \cdots, \alpha_{52}$ are not all zero, i' and I' are not both zero quantities. Hence $\rho'\neq 0$. But $\mathfrak{A}$ and $\mathfrak{B}$ are division algebras

* A theorem of A. Meyer, Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, vol. 29 (1884), pp. 209–222; see also Bachmann, *Zahlentheorie*, vol. IV$_1$, p. 266.

so that $i' \neq 0e$, $I' \neq 0E$. If we write $\rho' = \rho\pi^2$, where ρ and π are integers and ρ is a product of distinct primes, then, by replacing the $\alpha_{r2} (r = 1, \cdots, 5)$ by $\alpha_{r1} = \alpha_{r2}/\pi$, we replace i' by $i = i'/\pi$, I' by $I = I'/\pi$ and ρ' by ρ.

LEMMA 2. *There exist rational numbers $\alpha_{11}, \alpha_{21}, \cdots, \alpha_{51}$ with $\alpha_{11}, \alpha_{21}, \alpha_{31}$ not all zero and α_{41}, α_{51} not both zero, such that if*

$$i = \alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1, \quad I = \alpha_{41} I_1 + \alpha_{51} J_1, \tag{7}$$

then

$$i^2 = \rho e, \quad I^2 = \rho E, \quad \rho = \alpha_{41}^2 \rho_2 + \alpha_{51}^2 \sigma_2, \tag{8}$$

where ρ is a product of distinct rational prime integers.

Write $J = I_1 J_1$. Then obviously $JI = -IJ$ and $J^2 = \delta E$, δ in R. By replacing J by a rational multiple of itself we may obviously take δ a product of distinct primes. If $\alpha_{21} = \alpha_{31} = 0$, then by writing $j = j_1$, we have $ji = -ij$ and $j^2 = \gamma e$ with γ in R. If α_{21} and α_{31} are not both zero, then let

$$j = \alpha_{31} \rho_1 j_1 + \alpha_{21} i_1 j_1 \neq 0e, \tag{9}$$

whence $j^2 = -\sigma_1 \rho_1 (\alpha_{21}^2 - \alpha_{31}^2 \rho_1) e = \gamma e$ with γ in R.

Using the multiplication table of $\mathfrak{A}$, we find

$$\begin{aligned} ji &= (\alpha_{31}\rho_1 j_1 + \alpha_{21} i_1 j_1)(\alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1) \\ &= -\alpha_{31}\alpha_{11}\rho_1 i_1 j_1 + \alpha_{31}\alpha_{21}\rho_1\sigma_1 - \alpha_1 \rho_{31}^2 i_1 \\ &\quad - \alpha_{21}\alpha_{11}\rho_1 j_1 + \alpha_{21}^2 \sigma_1 i_1 - \alpha_{21}\alpha_{31}\rho_1\sigma_1 \\ &= (\alpha_{21}^2 \sigma_1 - \alpha_{31}^2 \rho_1) i_1 - \alpha_{11}\alpha_{21}\rho_1 j_1 - \alpha_{31}\alpha_{11}\rho_1 i_1 j_1, \end{aligned} \tag{10}$$

while

$$\begin{aligned} -ij &= -(\alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1)(\alpha_{31}\rho_1 j_1 + \alpha_{21} i_1 j_1) \\ &= -(\alpha_{11}\alpha_{31}\rho_1 i_1 j_1 + \alpha_{11}\alpha_{21}\rho_1 j_1 - \alpha_{21}^2 \sigma_1 i_1 + \alpha_{31}^2 \rho_1\sigma_1 i_1) . \end{aligned} \tag{11}$$

Hence $ji = -ij$ and in all cases j is not a polynomial in i with rational coefficients. It follows that e, i, j, ij are linearly independent with respect to R and form a basis of $\mathfrak{A}$. Similarly E, I, J, IJ are linearly independent with respect to R and form a basis of algebra $\mathfrak{B}$.

THEOREM 1. *By finding a single solution of a solvable diophantine equation we may represent any pair of generalized quaternion division algebras in the canonical form*

$$\mathfrak{A} = (e, i, j, ij),\ i^2 = \rho e,\ j^2 = \gamma e,\ ji = -ij, \tag{12}$$

$$\mathfrak{B} = (E, I, J, IJ),\ I^2 = \rho E,\ J^2 = \delta E,\ JI = -IJ, \tag{13}$$

with e and E respectively the moduli of $\mathfrak{A}$ and $\mathfrak{B}$, where ρ, γ and δ are multiplication constants expressed in terms of the original multiplication constants of $\mathfrak{A}$ and $\mathfrak{B}$ and the above solution, and where, without loss of generality, ρ, γ, δ may be taken to be products of distinct rational primes.

We shall now discuss a necessary and sufficient condition that any two generalized quaternion division algebras $\mathfrak{A}$ and $\mathfrak{B}$ be equivalent. We take the pair in the canonical form (12), (13). Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent so that there exists a (1-1) correspondence between the quantities of $\mathfrak{A}$ and $\mathfrak{B}$ which is preserved under addition, multiplication and scalar multiplication. The modulus e of $\mathfrak{A}$ will correspond to the modulus E of $\mathfrak{B}$. Let s in $\mathfrak{A}$ correspond to I, and t correspond to J. Then $s^2 = \rho e$, $t^2 = \delta e$ and $ts = -st$. But, by Lemma 1,

$$s = \lambda_1 i + \lambda_2 j + \lambda_3 ij \tag{14}$$

so that

$$s^2 = (\lambda_1^2 \rho + \lambda_2^2 \gamma - \lambda_3^2 \gamma\rho)e = \rho e, \tag{15}$$

and

$$\lambda_1^2 \rho + \lambda_2^2 \gamma - \lambda_3^2 \gamma\rho = \rho, \tag{16}$$

for rational $\lambda_1, \lambda_2, \lambda_3$. Let first λ_2 and λ_3 be not both zero. Then if

$$t_1 = \lambda_3 \rho j + \lambda_2 ij, \tag{17}$$

we have $t_1 s = -st_1$, $t_1 \neq 0e$, and

$$t_1^2 = \gamma[(\lambda_3\rho)^2 - \lambda_2^2 \rho]. \tag{18}$$

For we may evidently use here the proof (10), (11) by which we showed that $ji = -ij$. But $ts = -st$, so that $tst^{-1} = t_1 s t_1^{-1}$ and $(t_1^{-1}t)s(t_1^{-1}t)^{-1} = s$. It follows that $t_1^{-1}t$ is a polynomial in s and we may write

$$t = (\xi_1 e + \xi_2 s)t_1, \qquad (\xi_1 \text{ and } \xi_2 \text{ in } R). \tag{19}$$

Then $\delta e = t^2 = (\xi_1^2 - \xi_2^2\rho)t_1^2 = (\xi_1^2 - \xi_2^2\rho)[(\lambda_3\delta)^2 - \lambda_2^2\rho]\gamma e$. Write $\eta_1 = \xi_1(\lambda_3\rho) + \xi_2\lambda_2\rho, \eta_2 = \xi_1\lambda_2 + \xi_2(\lambda_3\rho)$, so that η_1 and η_2 are rational numbers. Then we have proved that, when λ_2 and λ_3 are not both zero,

$$\delta = (\eta_1^2 - \eta_2^2\rho)\gamma. \tag{20}$$

Next let $\lambda_2 = \lambda_3 = 0$. We have $s = \lambda_1 i$ and $\lambda_1^2\rho e = \rho e$ so that $\lambda_1 = 1$, $s = i$. We then take $t_1 = j$ and have $t_1^2 = \gamma e$. As before $t = (\xi_1 e + \xi_2 s)t_1$ and, if we write $\eta_1 = \xi_1$, $\eta_2 = \xi_2$, we again have (20).

Conversely let (20) be true. Then $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent under the correspondence

$$\mu_1 e + \mu_2 i + \mu_3 t + \mu_4 it \sim \mu_1 E + \mu_2 I + \mu_3 J + \mu_4 IJ, \tag{21}$$

where $\mu_1, \cdots, \mu_4$ are independent variables in R and $t = (\eta_1 e + \eta_2 i)j$. For $i^2 = \rho e$, $I^2 = \rho E$, $t^2 = \delta e$, $J^2 = \delta E$, $ti = -it$, $JI = -IJ$.

THEOREM 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two generalized quaternion division algebras over R. Then they are equivalent* if and only if, when they are put in a canonical form* (12), (13), *we have*

$$\delta = (\eta_1^2 - \eta_2^2\rho)\gamma \tag{20}$$

for rational η_1 and η_2.

Necessary and sufficient conditions that (20) be true are known. Let the greatest common divisor of γ and δ be ν so that $\gamma\delta = \nu^2\epsilon$, where ϵ is a product of distinct primes. Then (20) is true if and only if

$$\epsilon = \eta_3^2 - \eta_4^2\rho = 0 \tag{22}$$

for rational η_3, η_4. Let the greatest common divisor of ϵ and ρ be π so that $\epsilon = \epsilon'\pi$, $\rho = \rho'\pi$. Then (20) is true if and only if

$$\epsilon' = \pi\eta_5^2 - \rho'\eta_4^2 \tag{23}$$

* Any generalized quaternion algebra $\mathfrak{A}$ is self-reciprocal under the correspondence given by $i \sim i$, $j \sim j$, $ij \sim ji$. Hence we may add the word "reciprocal" to the above.

for rational η_3 and η_4, where now ϵ', π, ρ' and $\epsilon'\pi\rho'$ are each products of distinct primes. Then (20) is true if and only if

$$\pi\theta_1^2 - \rho'\theta_2^2 - \epsilon'\theta_3^2 = 0 \tag{24}$$

for integer $\theta_1, \theta_2, \theta_3$ not all zero. For when (20) is satisfied so is (24) when we write $\eta_3 = \theta_1/\theta_3$, $\eta_4 = \theta_2/\theta_3$ with integer $\theta_1, \theta_2, \theta_3$. Conversely let (24) be satisfied for integer $\theta_1, \theta_2, \theta_3$ not all zero. If $\theta_3 = 0$ then $\pi\theta_1^2 - \rho'\theta_2^2 = 0$, $\pi\rho' = \rho$, so that $(\pi\theta_1)^2 - \rho\theta_2^2 = 0$ contrary to the hypothesis that $\mathfrak{A}$ is a division algebra and hence ρ is not a rational square. It follows from our definitions of ϵ', π, ρ' that when (23) is satisfied so is (20). But (23) is satisfied by $\eta_3 = \theta_1/\theta_3$, $\eta_4 = \theta_2/\theta_3$. Hence (20) can be satisfied by rational η_1, η_2 if and only if the form in (24) is a null form. Using a known result of the theory of numbers*, we have the theorem:

THEOREM 3. *Let $\gamma\delta = \nu^2\epsilon$ where ν and ϵ are integers and ϵ is a product of distinct primes. Let the greatest common divisor of ϵ and ρ be π so that $\rho = \rho'\pi$, $\epsilon = \epsilon'\pi$. Then two division algebras $\mathfrak{A}$ and $\mathfrak{B}$ in a canonical form* (12), (13) *are equivalent if and only if*

$$-\epsilon'\rho' \text{ is a quadratic residue of } \pi, \tag{25}$$

$$\epsilon \text{ is a quadratic residue of } \rho', \tag{26}$$

$$\rho \text{ is a quadratic residue of } \epsilon'. \tag{27}$$

As a corollary of Theorem 2 we shall establish *the non-equivalence of any two of the D_τ algebras* of L. E. Dickson (loc. cit., Chapter IX) *which have different t's*. The D_τ algebras have $\rho = -1$ and $\gamma = \tau$ taken to be a product of distinct primes of the form $4n+3$. Let D_{τ_1} and D_{τ_2} be two such algebras with $\tau_1 \neq \tau_2$. Then $\tau_2 \neq (\eta_1^2 + \eta_2^2)\tau_1$ since otherwise we would have $\tau_2\tau_1$ expressible as a sum of two rational squares which is impossible when $\tau_2\tau_1$ contains a prime factor $4n+3$ to an odd power. Hence, by Theorem 2, D_{τ_1} and D_{τ_2} are non-equivalent.

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* See P. Bachmann, *Arithmetik der Quadratischen Formen*, Chapter 8.

GEODESIC COORDINATES OF ORDER r*

BY A. D. MICHAL

1. *Introduction.* Let $\Gamma^i_{\alpha\beta}$ be a general symmetric affine connection† and $\Gamma^i_{\beta_1\beta_2\ldots\beta_r}$, $(r=3, \cdots, p)$, the sequence of the first $p-2$ generalized symmetric affine connections defined by‡

$$\Gamma^i_{\alpha\beta\gamma} = \frac{1}{3} P\left(\frac{\partial \Gamma^i_{\alpha\beta}}{\partial x^\gamma} - 2\Gamma^i_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right)$$

and in general by the recurrence formula

$$\Gamma^i_{\alpha\beta\gamma\cdots\delta\epsilon} = \frac{1}{N} P\left(\frac{\partial \Gamma^i_{\alpha\beta\gamma\cdots\delta}}{\partial x^\epsilon} - \Gamma^i_{\sigma\beta\gamma\cdots\delta}\Gamma^\sigma_{\alpha\epsilon} - \cdots - \Gamma^i_{\alpha\beta\gamma\cdots\sigma}\Gamma^\sigma_{\delta\epsilon}\right).$$

The law of transformation of the affine connection $\Gamma^i_{\alpha\beta}$ is, as is well known,

$$\overline{\Gamma}^\sigma_{\alpha\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma} = \Gamma^i_{\lambda\mu}\frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta},$$

while the generalized affine connections transform in accordance with the law§

$$\overline{\Gamma}^\sigma_{\alpha\cdots\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma} = \frac{\partial^p x^i}{\partial \bar{x}^\alpha \cdots \partial \bar{x}^\beta} + \Gamma^i_{\lambda\cdots\mu}\frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\cdots\frac{\partial x^\mu}{\partial \bar{x}^\beta} + [\ \ ]$$

where the [ ] denotes the sum of terms, each of which involves a component $\Gamma^i_{\rho\ldots\sigma}$ with less than p subscripts, that vanish with $\Gamma^i_{\rho\alpha}, \cdots, \Gamma^i_{\rho\ldots\sigma\beta}$.

2. *Fundamental Theorems.*

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* Presented to the Society, December 30, 1929.

† We assume that the reader is conversant with the tensor theory as presented by O. Veblen in his *Invariants of Quadratic Differential Forms*, Cambridge Tract, 1927.

‡ O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25 (1923), p. 561.

§ T. Y. Thomas, American Journal Mathematics, vol. 50 (1928), p. 518.

THEOREM 1. *A necessary and sufficient condition that a coordinate system y^i determined by a coordinate system x^i and a point $x^i=q^i$ as origin have the property that $\Gamma^i_{\alpha\beta}$, $\Gamma^i_{\alpha\beta\gamma}$, $\cdots$, $\Gamma^i_{\alpha\beta\gamma\cdots\kappa}$ all vanish at the origin $y^i=0$ when evaluated in the y^i system of coordinates is that*

$$(1)\quad \begin{cases} \dfrac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = -\Gamma^i_{\lambda\mu}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial^p x^i}{\partial y^\alpha \partial y^\beta \partial y^\gamma \cdots \partial y^\delta} = -\Gamma^i_{\lambda\mu\cdots\rho}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta}\dfrac{\partial x^\nu}{\partial y^\gamma}\cdots\dfrac{\partial x^\rho}{\partial y^\delta}, \end{cases}$$

for $x^i=q^i$

If a star over a function denotes the evaluation of that function in the preferred system y^i, we obviously have

$$(2)\quad \begin{cases} {}^*\Gamma^\sigma_{\alpha\beta}(y)\dfrac{\partial x^i}{\partial y^\sigma} = \Gamma^i_{\lambda\mu}(x)\dfrac{\partial x^\lambda}{\partial y^\alpha}\dfrac{\partial x^\mu}{\partial y^\beta} + \dfrac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}^*\Gamma^\sigma_{\alpha\cdots\delta}\dfrac{\partial x^i}{\partial y^\sigma} = \Gamma^i_{\lambda\cdots\rho}\dfrac{\partial x^\lambda}{\partial y^\alpha}\cdots\dfrac{\partial x^\rho}{\partial y^\delta} + \dfrac{\partial^p x^i}{\partial y^\alpha \cdots \partial y^\delta} + [\ \ ] \end{cases}$$

where the [ ] on the right hand side of (2) denotes the sum of terms, each of which involves a component ${}^*\Gamma^i_{\rho\cdots\sigma}$ with less than p subscripts, that vanish with ${}^*\Gamma^i_{\rho\alpha}, \cdots, {}^*\Gamma^i_{\rho\cdots\beta}$. It is obvious now from (2) that a necessary and sufficient condition that

$$({}^*\Gamma^i_{\alpha\beta})_0 = 0,\ ({}^*\Gamma^i_{\alpha\beta\gamma})_0 = 0, \cdots,\ ({}^*\Gamma^i_{\alpha\beta\cdots\delta})_0 = 0$$

is that conditions (1) hold at the *origin* $x^i=q^i$.

DEFINITION. *A coordinate system y^i for which*

$$({}^*\Gamma^i_{\alpha\beta})_0 = 0,\ ({}^*\Gamma^i_{\alpha\beta_1\beta_2})_0 = 0, \cdots,\ ({}^*\Gamma^i_{\alpha\beta_1\beta_2\cdots\beta_r})_0 = 0$$

will be called a geodesic coordinate system of order r.

THEOREM 2. *A geodesic coordinate system y^i of order r for which*

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta_j^{\ i}$$

is defined implicitly by the coordinate transformation

$$(3)\quad \begin{cases} x^i = q^i + y^i - \dfrac{1}{2!}(\Gamma^i_{\alpha\beta}(x))_q y^\alpha y^\beta - \dfrac{1}{3!}(\Gamma^i_{\alpha\beta\gamma}(x))_q y^\alpha y^\beta y^\gamma - \cdots \\ \qquad - \dfrac{1}{(r+1)!}(\Gamma^i_{\alpha\beta_1\cdots\beta_r}(x))_q y^\alpha y^{\beta_1}\cdots y^{\beta_r} + \psi^i(y) \end{cases}$$

where ψ^i is a regular power series in y^i which begins with powers of y^i at least as great as $r+2$.*

Successive differentiations of (3) and evaluations for $y^i=0$ together with an application of Theorem 1 establishes our theorem.

From now on we shall deal exclusively with geodesic coordinate systems of the particular type considered in Theorem 2.

THEOREM 3. *If*

$$\bar{x}^i = f^i(x^1, \cdots, x^n)$$

is an arbitrary analytic transformation of the coordinates x^i, then any two geodesic coordinate systems of order r, y^i and $\bar{y}^i$, with the same origin $x^i=q^i$, that are determined by (3) *and*

$$\bar{x}^i = \bar{q}^i + \bar{y}^i - \frac{1}{2!}(\overline{\Gamma}^i_{\alpha\beta}(\bar{x}))_{\bar{q}}\bar{y}^\alpha\bar{y}^\beta - \cdots$$

$$- \frac{1}{(r+1)!}(\overline{\Gamma}^i_{\alpha\beta_1\cdots\beta_r}(\bar{x}))_{\bar{q}}\bar{y}^\alpha\bar{y}^{\beta_1}\cdots\bar{y}^{\beta_r} + \pi^i(\bar{y})$$

respectively, are related by the transformation

$$(4)\qquad \bar{y}^i = \left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_q y^\alpha + \lambda^i(y)$$

where λ^i is a power series in y^α that begins with powers of y^α not less than $r+2$.

To prove this theorem we observe first that

* We shall understand that $\psi^i\equiv 0$ is an admissible case.

$$\frac{\partial \bar{y}^i}{\partial y^\alpha} = \frac{\partial \bar{y}^i}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^k}\frac{\partial x^k}{\partial y^\alpha}. \tag{5}$$

On solving the equations (3) for y^i in the neighborhood of $x^i = q^i$ we obtain

$$y^i = x^i - q^i + h^i$$

where the h^i are power series in $x^i - q^i$ beginning with the second degree terms in $x^i - q^i$. Similarly

$$\bar{y}^i = \bar{x}^i - \bar{q}^i + k^i \tag{6}$$

where k^i involves terms in $\bar{x}^i - \bar{q}^i$ of higher order than the first. Hence, evaluating (5) at the common origin of the preferred coordinate systems y^i and $\bar{y}^i$, we obtain

$$\left(\frac{\partial \bar{y}^i}{\partial y^\alpha}\right)_0 = \left(\frac{\partial \bar{x}^i}{\partial x^\alpha}\right)_{\bar{q}} \tag{7}$$

on making use of the relations

$$\left(\frac{\partial x^i}{\partial y^\alpha}\right)_0 = \delta^i_\alpha, \qquad \left(\frac{\partial \bar{y}^i}{\partial \bar{x}^\alpha}\right) = \delta^i_\alpha.$$

By a similar method we also obtain the result

$$\left(\frac{\partial y^i}{\partial \bar{y}^\alpha}\right)_0 = \left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right)_{\bar{q}}.$$

The formulas that connect the components of the affine connection and the first $r-1$ generalized affine connections in the coordinate system y^i with those in the coordinate system $\bar{y}^i$ are

$$(8)\quad \begin{cases} {}^*\Gamma^\sigma_{\alpha\beta}\dfrac{\partial \bar{y}^i}{\partial y^\sigma} = {}^*\overline{\Gamma}^i_{\lambda\mu}\dfrac{\partial \bar{y}^\lambda}{\partial y^\alpha}\dfrac{\partial \bar{y}^\mu}{\partial y^\beta} + \dfrac{\partial^2 \bar{y}^i}{\partial y^\alpha \partial y^\beta}, \\ {}^*\Gamma^\sigma_{\alpha\cdots\delta}\dfrac{\partial \bar{y}^i}{\partial y^\sigma} = {}^*\overline{\Gamma}^i_{\lambda\cdots\mu}\dfrac{\partial \bar{y}^\lambda}{\partial y^\alpha}\cdots\dfrac{\partial \bar{y}^\mu}{\partial y^\delta} + \dfrac{\partial^p \bar{y}^i}{\partial y^\alpha \cdots \partial y^\delta} \\ \qquad\qquad + [\ ]\,(p = 3, 4, \cdots, r+1), \end{cases}$$

where the [ ] stands for a sum of terms that vanish with all the ${}^*\Gamma^\sigma_{\alpha\ldots\delta}$ in which the number of indices $\alpha, \cdots, \delta$ is less than p.

Evaluating (8) at the common origin of the coordinate systems y^i and $\bar{y}^i$ we obtain

$$\left(\frac{\partial^p \bar{y}^i}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_p}}\right)_0 = 0, \quad (p = 2, 3, \ldots, r+1). \tag{9}$$

Our theorem then follows immediately from this result and the relations (7).

3. *Applications.* Let $T^{a\cdots b}_{c\cdots d}(x)$ be the components of a tensor and ${}^*T^{a\cdots b}_{c\cdots d}(y)$ the components of the same tensor in a geodesic coordinate system y^i of order r with origin at $x^i = q^i$. A set of functions $T^{a\cdots b}_{c\cdots d,\, e\cdots f}$ of $x^1, \cdots, x^n$ at an arbitrarily given point $x^i = q^i$ will be defined by

$$(T^{a\cdots b}_{c\cdots d,\, e_1\cdots e_k}(x))_q = \left(\frac{\partial^k {}^*T^{a\cdots b}_{c\cdots d}(y)}{\partial y^{e_1} \cdots \partial y^{e_k}}\right)_0.$$

THEOREM 4. *The functions $T^{a\cdots b}_{c\cdots d,\, e_1\cdots e_k}(x^1, \cdots, x^n)$ for a fixed $k(\leqq r)$ are the components of a tensor.*

Let y^i and $\bar{y}^i$ be geodesic coordinates of order r with $x^i = q^i$ as their common origin. By hypothesis

$${}^*\overline{T}^{\alpha\cdots\beta}_{\gamma\cdots\delta}(\bar{y}) = {}^*T^{a\cdots b}_{c\cdots d}(y) \frac{\partial y^c}{\partial \bar{y}^\gamma} \cdots \frac{\partial y^d}{\partial \bar{y}^\delta} \frac{\partial \bar{y}^\alpha}{\partial y^a} \cdots \frac{\partial \bar{y}^\beta}{\partial y^b}.$$

Hence by differentiation and Theorem 3, we have

$$\frac{\partial^k {}^*\overline{T}^{\alpha\cdots\beta}_{\gamma\cdots\delta}(\bar{y})}{\partial \bar{y}^{e_1} \cdots \partial \bar{y}^{e_k}} = \frac{\partial^k {}^*T^{a\cdots b}_{c\cdots d}(y)}{\partial y^{e_1} \cdots \partial y^{e_k}} \frac{\partial y^c}{\partial \bar{y}^\gamma} \cdots \frac{\partial y^d}{\partial \bar{y}^\delta} \frac{\partial y^{e_1}}{\partial \bar{y}^{e_1}} \cdots \cdot \frac{\partial y^{e_k}}{\partial \bar{y}^{e_k}} \frac{\partial \bar{y}^\alpha}{\partial y^a} \cdots \frac{\partial \bar{y}^\beta}{\partial y^b} + [\ ] \tag{10}$$

where [ ] denotes the sum of a set of terms that vanish at the common origin of the coordinate systems y^i and $\bar{y}^i$. By Theorem 3, we see that

$$\left(\frac{\partial \bar{y}^i}{\partial y^j}\right)_0 = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_q, \qquad \left(\frac{\partial y^i}{\partial \bar{y}^j}\right)_0 = \left(\frac{\partial x^i}{\partial \bar{x}^j}\right)_{\bar{q}}.$$

If then we evaluate (10) at the origin and make use of the above relations in the resultant expression we obtain the law of transformation of a tensor evaluated at an arbitrarily given point. The proof of our theorem is therefore complete.

It is to be observed* that the tensor $T^{a\ldots b}_{c\ldots d.\ e}$ is the first affine extension (covariant derivative) of the tensor $T^{a\ldots b}_{c\ldots d}$ while $T^{a\ldots b}_{c\ldots d.\ e_1\ldots e_k}$ is the kth affine extension of $T^{a\ldots b}_{c\ldots d}$.

Similar results hold good for the extensions of the affine connection $\Gamma^{i}_{\alpha\beta}$. For example, the set of functions

$$\frac{\partial^{k*}\Gamma^{i}_{\alpha\beta}}{\partial y^{\gamma_1}\cdots\partial y^{\gamma_k}}, \qquad (k \leqq r),$$

evaluated at the origin of a geodesic coordinate system y^i of order $r+1$ define the kth *normal tensor* $A^{i}_{\alpha\beta\gamma_1\ldots\gamma_k}$. Thus the replacement theorems for affine differential invariants† of order r can be proved on the basis of geodesic coordinates of order $r+1$ without any hypothesis as to the existence of partial derivatives of the components of affine connection of higher order than the rth. Finally the replacement theorems for metric differential invariants‡ of order r can be proved on the basis of geodesic coordinates of order r without any hypothesis as to the existence of partial derivatives of the fundamental metric tensor $g_{\alpha\beta}$ of higher order than the rth.

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* The proofs of these statements are left for the reader since they are obtained by an obvious modification of Veblen's normal coordinate methods. See Veblen and Thomas, loc. cit., pp. 569–573.

† T. Y. Thomas and A. D. Michal, Annals of Mathematics, vol. 28 (1927), pp. 196–236

‡ T. Y. Thomas and A. D. Michal, Annals of Mathematics, vol. 28 (1927), pp. 631–688.

ON ALGEBRAIC SURFACES WHICH ARE INVARIANT IN A CERTAIN CLASS OF FINITE COLLINEATION GROUPS*

BY ARNOLD EMCH

1. *Introduction.* Among the algebraic varieties in projective space of 5 dimensions, S_5, and its sub-spaces and projections, there exist examples with notable properties and historic interest.

There is the Veronese surface of order 4 in S_5 obtained by a mapping process based on the ∞^5 conics of a plane. The projection of this surface from an S_1 which has no point in common with the surface upon an S_3 is a Steiner surface, as is well known. Again we have Segre's cubic variety cut out from the hypersurface

$$x_1^3 + x_2^3 + \cdots + x_6^3 = 0$$

by the hyperplane

$$x_1 + x_2 + \cdots + x_6 = 0$$

in S_5. The tangent hypercone from a generic point of this variety to the variety itself cuts a generic S_3 in a Kummer surface.

In what follows I shall consider a V_3 in S_5 defined as the intersection of two hyperquadrics in S_5. In coordinate representation this space shall be given by $S_5 \equiv (y_1, y_2, \cdots, y_6)$ and the two hyperquadrics by

$$y_1y_4 - y_2y_5 = 0, \tag{1}$$

$$y_2y_5 - y_3y_6 = 0. \tag{2}$$

Obviously the V_3 determined by (1) and (2) is also contained in the hyperquadric $y_1y_4 - y_3y_6 = 0$.

If to (1) and (2) the hyper n-ic

$$y_1^n + y_2^n + \cdots + y_6^n = 0 \tag{3}$$

* Presented to the Society, April 18, 1930.

is adjoined, then the three intersect in a surface which is of order $3n$ and may be represented by

$$(4) \quad y_2\, y_3^n(y_1^n + y_2^n + y_3^n + y_4^n) + y_1^n y_4^n(y_2^n + y_3^n) = 0.$$

The problem arises whether it is possible to reduce this surface birationally to a simpler form. This may indeed be accomplished by the following transformation of the space $S_5(y)$ into an ordinary space $S_3(x)$:

$$(5) \qquad \begin{aligned} \rho y_1 &= x_1x_4, \quad \rho y_2 = x_2x_4, \quad \rho y_3 = x_3x_4, \\ \rho y_4 &= x_2x_3, \quad \rho y_5 = x_3x_1, \quad \rho y_6 = x_1x_2. \end{aligned}$$

Clearly (1) and (2) are satisfied by (5). From this we find inversely

$$(6) \qquad \begin{aligned} \sigma x_1 &= 1/y_4, \qquad \sigma x_2 = 1/y_5, \qquad \sigma x_3 = 1/y_6, \\ \sigma x_4 &= (y_1 + y_2 + y_3)/(y_4y_5 + y_5y_6 + y_6y_4), \end{aligned}$$

with the adjunction of (1) and (2), so that the correspondence is of the Cremona type.

Since (5) is a Cremona transformation satisfying (1) and (2), and since the hyper n-ic (3) is transformed by (5) into the surface

$$(7) \quad x_1^n x_2^n + x_2^n x_3^n + x_3^n x_1^n + x_1^n x_4^n + x_2^n x_4^n + x_3^n x_4^n = 0,$$

an F_{2n} of S_3, the surface (4) defined by (1), (2) and (3) in S_5 is transformed birationally by (5) into the surface (7).

The problem is now resolved into an investigation of these surfaces F_{2n} of order $2n$ in three dimensions.

2. *Group Properties of Surface F_{2n}.* As (7) is symmetric in the x's, the surface is obviously invariant under the symmetric G_{24} on four letters. But there exists a larger group of collineations containing the symmetric group as a subgroup. Denote the n roots of unity by $1, r_2, r_3, \cdots, r_n$. Then as $(r_k x_i)^n = x_i^n$, it is clear that

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_{1r_\alpha} & x_{2r_\beta} & x_{3r_\gamma} & x_{4r_\delta} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_{2r_i} & x_{3r_k} & x_{4r_l} \end{pmatrix}$$

a substitution which leaves F_{2n} invariant. There are as many such substitutions as there are dispositions of n elements with

repetitions in 3 places, that is, n^3. Each of these can be permuted in 24 different ways, so that there exists a collineation group of order $24 \cdot n^3$ under which the surface is invariant. When n is even, the octavic group

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ \pm x_1 & \pm x_2 & \pm x_3 & \pm x_4 \end{pmatrix}$$

may be adjoined, so that in this case the surface is invariant under a group of order $8 \cdot 24 \cdot n^3$. The group of n^3 collineations is Abelian and is invariant in the entire group, since the transform of any of its substitutions

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_{2r_i} & x_{3r_k} & x_{4r_l} \end{pmatrix} \text{ by } \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_a & x_b & x_c & x_d \end{pmatrix}$$

is

$$\begin{pmatrix} x_a & x_b & x_c & x_d \\ x_a & x_{br_i} & x_{cr_k} & x_{dr_l} \end{pmatrix}.$$

The entire group is the direct product of the symmetric group on four elements and this Abelian invariant subgroup. The result may be stated as follows:

THEOREM 1. *The symmetric surface* $\sum x_i^n x_k = 0$, $i \neq k$, *is invariant under a collineation group of order* $24 \cdot n^3$ *or* $8 \cdot 24 \cdot n^3$, *according as n is odd or even. The group is the direct product of the symmetric group* G_{24}, *or a group* G_{192} *in case of an even n, with an invariant Abelian subgroup of order* n^3.

3. *Conics on the Surface.* The surface may be written in the form

$$(x_2^n + x_3^n)(x_1^n + x_4^n) + x_1^n x_4^n + x_2^n x_3^n = 0$$

or

$$\Pi(x_2 - e^{(1+2k)i\pi/n} \cdot x_3) \cdot \Pi(x_1 - e^{(1+2k)i\pi/n} \cdot x_4) + \Pi(x_1 x_4 - e^{(1+2k)i\pi/n} \cdot x_2 x_3) = 0;\ k = 0, 1, \cdots, n-1.$$

The first product, set equal to zero, represents n planes through the side A_1A_4 of the coordinate tetrahedron. The third product represents n quadrics on the skew quadrilateral A_1A_2, A_2A_4, A_4A_3, A_3A_1. Hence there are n planes through the side A_1A_4 which cut these quadrics in n^2 conics which lie on the surface F_{2n}. The same is, of course, true for every other side of the coordinate tetrahedron. In each of these planes there are n

of these conics. When n is odd, $n=2m+1$, then for $k=m$ the corresponding plane through A_1A_4 becomes $x_2+x_3=0$, which cuts the quadric $x_1x_4+x_2x_3=0$ in a real conic. Hence, when n is odd, the surface contains 6 real conics which lie on the quadric $\sum x_ix_k=0$, $i\neq k$. In this case the surface also passes through the points $(-1, 1, 1, 1)$; $(1, -1, 1, 1)$; $(1, 1, -1, 1)$; $(1, 1, 1, -1)$ which form a desmic tetratredron with respect to the coordinate tetrahedron. The four centers of perspective of these form the third desmic tetrahedron $(1, 1, 1, 1)$; $(-1, -1, 1, 1)$; $(-1, 1, -1, 1)$; $(1, -1, -1, 1)$, which is not on the surface. The points A_1, A_2, A_3, A_4 are obviously n-fold points of the surface. Hence we have the following theorem.

THEOREM 2. *The surface F_{2n} has A_1, A_2, A_3, A_4 as n-fold points and contains $6n^2$ conics which lie 6 times on n^2 planes through the joins A_iA_k. On each of these planes lie n of the conics. When n is odd, then 6 of these conics are real and lie on the symmetric quadric through the four multiple points.*

4. *Lines on the Surface.* If the equation of F_{2n} is written in the form

$$x_1^n x_2^n + x_2^n x_3^n + x_3^n x_1^n + x_4^n(x_1^n + x_2^n + x_3^n) = 0$$

it is clear that the $2n^2$ lines of intersection of the two cones $x_1^nx_2^n + x_2^nx_3^n+x_3^nx_1^n=0$, $x_1^n+x_2^n+x_3^n=0$ lie on the surface. Thus is established the existence of $8n^2$ lines, $2n^2$ through each multiple point, on the surface. The equations of these lines may easily be established. Setting $x_1^n=z_1$, $x_2^n=z_2$, $x_3^n=z_3$, the traces of the lines through A_4 for example are obtained as the solutions of $z_1z_2+z_2z_3+z_3z_1=0$ and $z_1+z_2+z_3=0$ or of $x_1^n=1$, $x_2^n=\omega$, $x_3^n=\omega^2$, where 1, ω, ω^2 are the cube roots of unity. These solutions may be written in the form

$$x_1 = e^{2ki\pi/n},\quad x_2 = e^{2i\pi/(3n)+2ki\pi/n},\quad x_3 = e^{4i\pi/(3n)+2ki\pi/n},$$

or

$$x_1 = 1,\quad x_2 = e^{2i\pi[3n+3(k-k_1)+1]/(3n)},\quad x_2 = e^{2i\pi[3n+3(k-k_1)+2k]/(3n)},$$

$k, k_1=0, 1, \cdots, n-1$, hence also $k-k_1=0, \pm1, \cdots, \pm(n-1)$. Now we may distinguish three cases, according as $n\equiv0, 1, 2 \pmod 3$, or $n=3m$, $n=3m+1$, $n=3m+2$. When $n=3m$, then it is impossible for x_2 or x_3 to become either ω, or ω^2, since $6m+3(k-k_1)+1\neq0$ and also $6m+3(k-k_1)+2\neq0$. This however is possible when $n=3m+1$, by choosing $k_1=k=2m+1$, to

get $x_2=\omega$, and $k_1-k=m+1$ to get $x_3=\omega^2$. Likewise when $n=3m+2$, for $k_1-k=m+1$, we have $x_2=\omega^2$, and for $k_1-k=2(m+1)$, $x_3=\omega$. Now $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are the intersections of the unit line $x_1+x_2+x_3=0$ with the conic $x_1x_2+x_2x_3+x_3x_1=0$, so that when $n=3m+1$, or $3m+2$, two of the $2n^2$ lines through A_4 pass through these points in $x_4=0$. Two such lines pass through each of the remaining three vertices. The six traces of these lines lie on the unit-plane $x_1+x_2+x_3+x_4=0$ and are the intersections of the quadric $\sum x_ix_k=0$, $i\neq k$, with the quadrilateral cut out on the unit-plane by the faces of the coordinate tetrahedron. But when $n=3m$, there are $18m^2$ lines through each vertex whose traces on the opposite face lie by sixes on $3m^2$ conics; i.e., the lines lie by sextuples on $3m^2$ quadric cones with the same vertex.

These lines intersect by pairs in the following manner: Let $S=(1, s_1, s_2)$ be one of the $2n^2$ solutions as represented above by (x_1, x_2, x_3) in terms of exponentials. Then $(1, s_1, s_2, s_3)$ is a point on the line A_4S which lies on the surface. If we now let s_3 take on the $3n$ values $1, r_2, \cdots, r_n; s_1^1, s_1^2, \cdots, s_1^n; s_2^1, s_2^2, \cdots, s_2^n$ corresponding to the $3n$ solutions for $x_1^n=1$, $x_2^n=\omega$, $x_3^n=\omega^2$ separately, we obtain $3n$ points on A_4S through each of which pass two lines of the surface. For example through $(1, s_1, s_2, s_1^i)$ pass the lines joining A_4 to $(1, s_1, s_2, 0)$ and A_2 to $(1, 0, s, s_1^i)$. Thus there exist $3n$ lines of the surface which intersect a line A_4S of the surface, such that through each vertex A_1, A_2, A_3 pass n of the 3 lines. Hence on account of the symmetry there exist $\frac{1}{2}\cdot 3n\cdot 2n^2\cdot 4=12n^3$ points on the surface through each of which pass two lines of the surface. This may be summed up as follows.

THEOREM 3. *There exist $8n^2$ lines on the surface, $2n^2$ through each multiple point. These lines intersect by pairs in $12n^3$ points of which $3n$ lie on each line. In case $n=3m$, the lines through each point A_i lie by sixes on $3m^2$ cones with the common vertex A_i. In case $n=3m+1$ there are $3m^2-2m$, in case $n=3m+2$, there are $3m^2+4m+2$ such cones. The configuration of $12n^3$ points is, of course, invariant under the finite collineation group of the surface.*

The surface F_{2n} is also invariant under the involutorial cubic transformation

$$\rho x_1' = x_2x_3x_4, \quad \rho x_2' = x_1x_3x_4, \quad \rho x_3' = x_1x_2x_4, \quad \rho x_4' = x_1x_2x_3.$$

From this it follows *that F_{2n} is invariant in a group of Cremona transformations whose order is twice the order of the corresponding collineation group.*

5. *Example. The Sextic Surface* $\sum x_i^2x_k^2 = 0$, $i \neq k$. This surface assumes a particularly attractive form by the following transformation from projective into Cartesian coordinates. Choose a cube $A_1(-1, 1, 1)$; $A_2(1, -1, 1)$; $A_3(1, 1, -1)$; $A_4(-1, -1, -1)$; $A_1'(1, -1, -1)$; $A_2'(-1, 1, -1)$; $A_3'(-1, -1, 1)$; $A_4'(1, 1, 1)$, so that the Cartesian axes OX, OY, OZ are the lines joining the centers of opposite faces of the cube. Now let the coordinate tetrahedron of the projective space (x_1, x_2, x_3, x_4) coincide with $A_1A_2A_3A_4$; then

$$\rho x_1 = -x + y + z + 1, \qquad \rho x_2 = x - y + z + 1,$$
$$\rho x_3 = x + y - z + 1, \qquad \rho x_4 = -x - y - z + 1.$$

The surface F_6 reduces to

$$\begin{aligned}(-x+y+z+1)^3(x-y+z+1)^3 &+ (x-y+z+1)^3(x+y-z+1)^3 \\ &+ (x+y-z+1)^3(-x+y+z+1)^3 \\ &+ (-x-y-z+1)^3\{(-x+y+z+1)^3 \\ &+ (x-y+z+1)^3 + (x+y-z+1)^3\} = 0.\end{aligned}$$

The quadric $\sum x_ix_k = 0$, $i \neq k$, reduces to the sphere $x^2+y^2+z^2-3=0$, so that *the six real conics on the sextic surface become circles* which lie in the faces of the cube and which in each case pass through the four vertices of the cube in this face. The unit plane is transformed into the plane at infinity and the three desmic tetrahedra become the two regular tetrahedra inscribed in the cube and the tetrahedron formed by the origin and the infinite points of the x-, y-, and z-axis. There are 72 lines on the sextic. *The 18 lines through each vertex lie by sixes on 3 coaxial right circular cones with the principal diagonals of the cube as axes. The collineation group under which the surface is invariant is of order $24 \cdot 27 = 648$. It also permutes the points of the configuration of 324 points through each of which pass 2 of the 72 lines. On each line there are 9 of these points.*

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NEW CRITERIA ASSOCIATED WITH FERMAT'S LAST THEOREM*

BY JOHN MCDONNELL

Furtwängler has obtained† by means of Eisenstein's law of reciprocity for residues of pth powers, p an odd prime, certain criteria in connection with the solution of the equation

$$x^p + y^p + z^p = 0, \tag{1}$$

where x, y, z are relatively prime rational integers, and these criteria involve the rational factors of x, y, z, $y-z$, $z-x$, $x-y$.

It is the object of the present article to employ the same method to derive similar criteria for the factors of

$$x^2 - yz,\ y^2 - zx,\ z^2 - xy,\ x^2 + yz,\ y^2 + zx,\ z^2 + xy.$$

THEOREM 1. *If x, y, z satisfy equation* (1), *$yz+zx+xy$ is prime to p, and r is any factor of x^2-yz, then $r^{p-1}\equiv 1 \pmod{p^2}$.*

PROOF. Let α be a pth root of unity. We have from the identity

$$x(x + y\alpha) - y(z + x\alpha) = x^2 - yz,$$

the equation in pth power characters

$$\left\{\frac{x(x + y\alpha)}{r}\right\} = \left\{\frac{y(z + x\alpha)}{r}\right\}$$

or, since

$$\left\{\frac{x}{r}\right\} = \left\{\frac{y}{r}\right\} = 1,$$

x and y being rational, we have

$$\left\{\frac{x + y\alpha}{r}\right\} = \left\{\frac{z + x\alpha}{r}\right\}. \tag{2}$$

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* Presented to the Society, August 29, 1929.

† Wiener Sitzungsberichte, vol. 121, IIa (1912), pp. 589–592.

We shall now obtain further relations between the above characters from the theory developed in connection with Fermat's theorem. One, at least, of the numbers y, z must be prime to p. Assuming, first, that z is prime to p, we have from equation (1) the following well known result

$$x + y + z \equiv 0 \pmod{p}.$$

Moreover, the ideal $(x+y\alpha)$ is the pth power of an ideal factor of z: the number $x+y\alpha$ is prime to $1+\alpha$, and is, consequently, associated with a semi-primary number of the form $\alpha^n (x+y\alpha)$, where n is determined by the congruence

$$nz \equiv y \pmod{p}.$$

By Eisenstein's reciprocity theorem, we have*

$$\left\{\frac{\alpha^n(x + y\alpha)}{r}\right\} = \left\{\frac{r}{\alpha^n(x + y\alpha)}\right\} = 1. \tag{3}$$

Again if y be prime to p we obtain similarly

$$\left\{\frac{\alpha^m(z + x\alpha)}{r}\right\} = 1 \tag{4}$$

with $my \equiv x \pmod{p}$.

If, now, yz be prime to p both equations (3) and (4) are satisfied and together with (2) yield the result

$$\left\{\frac{\alpha}{r}\right\}^n = \left\{\frac{\alpha}{r}\right\}^m,$$

or

$$\left\{\frac{\alpha}{r}\right\}^{m-n} = 1. \tag{5}$$

From the congruences $nz \equiv y$ and $my \equiv x \pmod{p}$, we have

$$(m - n)yz \equiv zx - y^2.$$

Now

$$zx - y^2 = (yz + zx + xy) - y(x + y + z),$$

$$x + y + z \equiv 0 \pmod{p},$$

$$yz + zx + xy \not\equiv 0.$$

* Furtwängler, loc. cit.

Hence $m-n \not\equiv 0$ and, therefore, from (5) we have*

$$\left\{\frac{\alpha}{r}\right\} = 1, \text{ and } r^{p-1} \equiv 1 \pmod{p^2}. \tag{6}$$

On the other hand, if yz be not prime to p, either y or z is divisible by p. Assuming first that z is divisible, we have, from equation (1), the following results:

$$x + y \equiv 0 \pmod{p^2},$$

$$x + y\alpha \text{ is divisible by } 1 - \alpha.$$

Denoting the quotient $(x+y\alpha)/(1-\alpha)$ by ω, we see that the principal ideal (ω) is the pth power of an ideal factor of z and, consequently,

$$\left\{\frac{r}{\omega}\right\} = 1.$$

Furthermore

$$\omega = -y + (x + y)/(1 - \alpha),$$

and $x+y\equiv 0 \pmod{p^2}$; hence ω is a primary integer, and by Eisenstein's theorem,

$$\left\{\frac{\omega}{r}\right\} = \left\{\frac{r}{\omega}\right\} = 1.$$

Since $x+y\alpha = (1-\alpha)\omega$, it follows that

$$\left\{\frac{x + y\alpha}{r}\right\} = \left\{\frac{1 - \alpha}{r}\right\}.$$

Again y is prime to p; consequently, from (4),

$$\left\{\frac{\alpha^{-1}(z + x\alpha)}{r}\right\} = 1, \quad m \equiv -1, \pmod{p},$$

or

$$\left\{\frac{z + x\alpha}{r}\right\} = \left\{\frac{\alpha}{r}\right\}.$$

* Furtwängler, loc. cit

Substituting in (2), we have

$$\left\{\frac{1-\alpha}{r}\right\} = \left\{\frac{\alpha}{r}\right\} \tag{7}$$

or, squaring,

$$\left\{\frac{1-2\alpha+\alpha^2}{r}\right\} = \left\{\frac{\alpha}{r}\right\}^2,$$

or

$$\left\{\frac{\alpha}{r}\right\}\left\{\frac{\alpha^{-1}-2+\alpha}{r}\right\} = \left\{\frac{\alpha}{r}\right\}^2.$$

Dividing both sides by $\{\alpha/r\}$, and noting that

$$\left\{\frac{\alpha^{-1}-2+\alpha}{r}\right\} = 1,$$

since $\alpha^{-1}-2+\alpha$ is a real number, we obtain,

$$\left\{\frac{\alpha}{r}\right\} = 1,$$

and hence from (6)

$$r^{p-1} \equiv 1 \pmod{p^2}.$$

A similar proof obviously holds when y is divisible by p.

THEOREM 2. *If x, y, z satisfy equation* (1), $x(y-z)(x^2+yz)$ *is prime to p, and r is any factor of x^2+yz, then*

$$r^{p-1} \equiv 1 \pmod{p^2}.$$

PROOF. From the identity

$$(x+y\alpha)(x+z\alpha^{-1}) = x^2+yz+x(y\alpha+z\alpha^{-1}),$$

we readily obtain the equation

$$\left\{\frac{x+y\alpha}{r}\right\}\left\{\frac{x+z\alpha^{-1}}{r}\right\} = \left\{\frac{y\alpha+z\alpha^{-1}}{r}\right\}. \tag{8}$$

Assuming, first, that yz is prime to p and remembering that x is so by hypothesis, we see that $y\alpha+z\alpha^{-1}$, $x+z\alpha^{-1}$, $x+y\alpha$

are all prime to $1-\alpha$ and are, consequently, associated with semi-primary integers of the form $\alpha^l(y\alpha+z\alpha^{-1})$, $\alpha^m(x+z\alpha^{-1})$, $\alpha^n(x+y\alpha)$ respectively. From the reasoning developed in the previous theorem it is easily seen that

$$\left\{\frac{\alpha^l(y\alpha+z\alpha^{-1})}{r}\right\} = \left\{\frac{\alpha^m(x+z\alpha^{-1})}{r}\right\} = \left\{\frac{\alpha^n(x+y\alpha)}{r}\right\} = 1;$$

hence, from (8), we have

$$\left\{\frac{\alpha}{r}\right\}^{m+n-l} = 1.$$

Now the congruences satisfied by l, m, n, namely

$$lx \equiv y - z, \quad my \equiv -z, \quad nz \equiv y \pmod{p},$$

yield the congruence

$$\begin{aligned}(m+n-l)xyz &\equiv -z^2x + xy^2 - yz(y-z),\\ &\equiv (y-z)[x(y+z) - yz] \pmod{p}\\ &\equiv -(y-z)(x^2+yz),\\ &\not\equiv 0, \text{ by hypothesis.}\end{aligned}$$

Hence

$$\left\{\frac{\alpha}{r}\right\} = 1,$$

and from (6)

$$r^{p-1} \equiv 1 \pmod{p^2}.$$

On the other hand if yz is not prime to p, either y or z must be divisible by p. Assuming z to be divisible and recalling the arguments in the corresponding case in Theorem I, we have immediately the following relations:

$$\left\{\frac{x+y\alpha}{r}\right\} = \left\{\frac{1-\alpha}{r}\right\}, \quad \left\{\frac{x+z\alpha^{-1}}{r}\right\} = 1,$$

and

$$\left\{\frac{y\alpha+z\alpha^{-1}}{r}\right\} = \left\{\frac{\alpha}{r}\right\}\left\{\frac{y+z\alpha^{-2}}{r}\right\} = \left\{\frac{\alpha}{r}\right\}.$$

From the above and from (8), we obtain

$$\left\{\frac{1-\alpha}{r}\right\} = \left\{\frac{\alpha}{r}\right\},$$

and from (7) it follows that

$$r^{p-1} \equiv 1 \pmod{p^2}.$$

As before, a similar proof obtains when y is divisible by p.

OTTAWA, CANADA

ON THE SOLUTION OF THE EULER EQUATIONS FOR THEIR HIGHEST DERIVATIVES*

BY H. V. CRAIG

1. *Introduction.* J. H. Taylor† has given two elegant methods of solving for their highest derivatives the Euler equations associated with the integral $\int F(x, \dot{x})dt$. In this paper these two methods are modified so as to apply to the more general case in which the Euler equations contain derivatives of order higher than the second.

2. *Notation.* Throughout this paper we shall employ vector notation and shall use dots and enclosed superscripts to indicate differentiation with respect to the parameter. Thus x, $\dot{x}$, $x^{(m)}$ will stand for the sets

$$x^1, x^2, \cdots, x^n; \quad \frac{dx^1}{dt}, \frac{dx^2}{dt}, \cdots, \frac{dx^n}{dt}; \quad \frac{d^m x^1}{dt^m}, \frac{d^m x^2}{dt^m}, \cdots, \frac{d^m x^n}{dt^m},$$

respectively. Partial derivatives will be denoted by means of subscripts, thus

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* Presented to the Society, September 7, 1928. This paper is a part of a thesis written at the University of Wisconsin under the direction of Professor J. H. Taylor.

† J. H. Taylor, *The reduction of Euler's equations to a canonical form*, this Bulletin, vol. 31 (1925) p. 257.

$$\frac{\partial F(x,\dot{x}, \cdots, x^{(m)})}{\partial x^i} = F_i; \quad \frac{\partial F(x,\dot{x}, \cdots, x^{(m)})}{\partial \dot{x}^i} = F_{i^1};$$

$$\frac{\partial F(x,\dot{x}, \cdots, x^{(m)})}{\partial \ddot{x}^i} = F_{i^2}; \cdots.$$

However, if the differentiation is with respect to the highest derivatives present we shall further abbreviate by omitting the m, that is,

$$\frac{\partial F(x,\dot{x}, \cdots, x^{(m)})}{\partial x^{i(m)}} = F_i.$$

Summations are to be understood when repeated indices occur.

3. *The Calculus of Variations Problem and the Euler Equations.* We consider a function $F(x, \dot{x}, \cdots, x^{(m)})$, $m>1$, with properties to be specified and seek among all curves of class $2m$ lying in a certain region of an n-space and satisfying certain boundary conditions, the one which gives the integral

$$I = \int_{t_1}^{t_2} F(x,\dot{x}, \cdots, x^{(m)})dt$$

its minimum value.

As a first hypothesis on F we suppose, as just implied, that the solution of this problem exists uniquely. The additional hypotheses are: (a), that F is of class $m+1$; (b), that the classical $\overline{F}_1$ function* associated with F does not vanish along the solution; and (c), that I is independent of the choice of the parameter.

Zermelo has shown† that this invariance of I implies the following identities in $x, \dot{x}, \cdots, x^{(m)}$:

(1) $\qquad \dot{x}^i F_i \equiv 0, \quad i$ ranges from 1 to n,

(2) $\qquad x^{i(\alpha)} E_{\alpha i} \equiv F, \quad \alpha$ ranges from 1 to m,

* We have used $\overline{F}_1$ to avoid confusion with $F_1 = \partial F/\partial x^{1(m)}$. For a discussion of the $\overline{F}_1$ function, see Oscar Bolza, *Vorlesungen über Variationsrechnung*, Leipzig, Teubner, 1909, p. 13. Oscar Bolza, *Lehrbuch der Variationsrechnung*, p. 196.

† See Adolph Kneser, *Lehrbuch der Variationsrechnung*, Leipzig, Teubner, 1925, p. 217.

where E_{qi} is defined by the equation

$$E_{qi} \equiv (-1)^{\beta} F^{\beta}{}_{iq+\beta}, \quad (\beta \text{ ranges from 0 to } m-q).$$

We observe from this definition that $E_{0i}=0$, $i=1, 2, \cdots, n$, are the Euler equations associated with F. These equations are not independent, but satisfy the relation

$$\dot{x}^i E_{0i} = 0. \tag{3}$$

This may be established as follows. Differentiating (2) we obtain

$$\dot{F} = x^{i(\alpha)} \dot{E}_{\alpha i} + x^{i(\alpha+1)} E_{\alpha i}.$$

If in this we replace the second term by its value as given by the formula

$$E_{qi} = F_{iq} - \dot{E}_{q+1\,i}$$

the relation becomes

$$\dot{F} = x^{i(\alpha)} \left[F_i{}^{\alpha-1} - E_{\alpha-1\,i} \right] + x^{i(\alpha+1)} E_{\alpha i}.$$

Since $E_{mi}=F_i$, this may finally be written

$$\dot{F} = x^{i(\alpha)} F_i{}^{\alpha-1} + x^{i(m+1)} F_i - \dot{x}^i E_{0i} + x^{i(\alpha)} E_{\alpha-1i} - x^{i(\alpha)} E_{\alpha-1i};$$

hence $\dot{x}^i E_{0i}=0$.

Since (1) evidently leads to

$$\dot{x}^i F_{ij} = 0, \qquad (j = 1, 2, \cdots, n), \tag{4}$$

it follows that the determinant $|F_{ij}|$ (and this is the determinant of the coefficients of $x^{(2m)}$ in the Euler equations) vanishes. Accordingly, the problem of solving the Euler equations for their highest derivatives requires special consideration. We shall make it a part of our hypotheses on F that the rank of this determinant be $n-1$.

For use in determining the rank of certain determinants which will appear presently, we insert here a few miscellaneous observations. As a consequence of equation (4) and the rank of the determinant $|F_{ij}|$, the cofactors F^{ij} of the latter satisfy the following relations:

$$\frac{\dot{x}^1}{F^{i1}} = \frac{\dot{x}^2}{F^{i2}} = \cdots = \frac{\dot{x}^n}{F^{in}}.$$

If we note that the quantities F^{ii} are symmetric in their indices, these equalities are seen to be expressible in the form

$$\frac{\dot{x}^i\dot{x}^j}{F^{ij}} = \frac{\dot{x}^k\dot{x}^l}{F^{kl}}, \tag{5}$$

where i, j, k, l may each be any number of the set $1, 2, \cdots, n$ and no summation is to be understood. The reciprocal of the common value of the members of (5) is the $\overline{F}_1$ function of our problem.

4. *The First Method of Solution.** Let $H(x, \dot{x})$ be any function of class $2m-1$, homogeneous of degree plus one in $\dot{x}$, and non-vanishing along the solution of our problem. With these restrictions we may so select the parameter that H will maintain the value unity along the solution. Differentiating the equation $H=1$, $2m-1$ times with respect to t, yields the relation

$$x^{i(2m)}H_i + r = 0. \tag{6}$$

Here we have written explicitly only the terms in $x^{(2m)}$ and have represented by r the remaining terms. This equation we adjoin to the system

$$x^{i(2m)}F_{ij} + wH_j + R_j = 0, \tag{7}$$

R_j being so chosen that these relations reduce to the Euler equations for $w=0$. The system (6, 7) is linear in the variables $x^{(2m)}$ and w, has a non-vanishing determinant, and determines the same set of values for $x^{(2m)}$ as the Euler equations. To prove this last statement we multiply the equations of the set (7) by $\dot{x}^j$ and sum. Because of (3) and the conditions imposed on H (the homogeneity of H implies $\dot{x}^iH_i=H$) the result is $w=0$. The determinant of the system (6, 7) is equal to

$$\pm H_iH_jF^{ij} = \pm \overline{F}_1\dot{x}^i\dot{x}^jH_iH_j = \pm \overline{F}_1 \neq 0.$$

5. *The Second Method of Solution.* Let us replace the function $F(x, \dot{x}, \cdots, x^{(m)})$ of our calculus of variations problem with a new function $f(x, \dot{x}, \cdots, x^{(m)})$, which we define as follows:

* Taylor points out the incidence of this method in an article entitled *The properties of curves in space which minimize a definite integral*, by Mason and Bliss, Transactions of this Society, vol. 9 (1908), p. 443.

$$f(x,\dot{x}, \cdots, x^{(m)}) \equiv F(x,\dot{x}, \cdots, x^{(m)}) + \frac{1}{2}\left\{\frac{d^{m-1}H(x,\dot{x})}{dt^{m-1}}\right\}^2.$$

We restrict the function H as in the preceding section and select the parameter as before so that H maintains the value unity along the unique solution C.

For such a parameter it is evident that

$$\int_C f(x,\dot{x}, \cdots, x^{(m)})dt = \int_C F(x,\dot{x}, \cdots, x^{(m)})dt,$$

and

$$\int_{\bar{C}} fdt > \int_C Fdt,$$

if $\bar{C}$ is any other admissible curve. Therefore the curve C is also an extremal of the integral $\int fdt$. (The problem associated with f is not a Weierstrass problem since t has a special meaning.) Furthermore the determinant $|f_{ij}|$ $(f_{ij}=F_{ij}+H_iH_j)$ is different from zero since

$$|f_{ij}| = H_iH_jF^{ij} = \overline{F}_1\dot{x}^i\dot{x}^jH_iH_j \neq 0.*$$

Hence the values of the quantities $x^{(2m)}$ along any extremal may be obtained by solving the Euler equations associated with f by Cramer's rule.

It can be shown that the left members of the Euler equations in the unsolved form are the components of a covariant vector. The method of solution outlined above gives us a simple contravariant description of this vector.

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* See J. H. Taylor, loc. cit. p. 261, for the development of a similar determinant.

A NOTE ON A SYSTEM OF EQUATIONS WITH INFINITELY MANY UNKNOWNS*

BY L. W. COHEN†

If the system of equations

$$x_i + \sum_{k=1}^{\infty} a_{ik}x_k = y_i \tag{1}$$

is considered as a transformation on the space (x_k) such that $\sum_{k=1}^{\infty}|x_k|^p$ converges for some $p>1$, then we must require that $\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}$ converges for every i.‡ If we further require that this transformation carry (x_k) into (y_k) of the same space, we are led to consider the inequality

$$\left[\sum_{i=1}^{\infty}|y_i|^p\right]^{1/p} \leqq \left[\sum_{i=1}^{\infty}|x_i|^p\right]^{1/p} \cdot \left[1+\left\{\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}\right]^{p-1}\right\}^{1/p}\right] \tag{2}$$

obtained from (1) by the Hölder and Minkowski inequalities.§ The convergence of the double series on the right is a sufficient condition. Since p and $p/(p-1)$ are separated by 2 if not equal to it, we will consider $p\leqq 2$. It will appear in IV below that in the case $p\geqq 2$, the matrix of the transformation is the transposed of that of (1). For the system (1), under the condition that

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‡ Landau, *Über einen Konvergentzsatz*, Göttinger Nachrichten, 1907, pp. 25–27.

§ These are respectively

$$\sum|a_nb_n| \leqq \left[\sum|a_n|^p\right]^{1/p}\left[\sum|b_n|^{p/(p-1)}\right]^{(p-1)/p}$$

and

$$\left[\sum|a_n+b_n|^p\right]^{1/p} \leqq \left[\sum|a_n|^p\right]^{1/p}+\left[\sum|b_n|^p\right]^{1/p}, \qquad (p>1).$$

See F. Riesz, *Les Systèmes d'Équations Linéaires* · · · , Paris, 1913, p. 45.

$$(3)\qquad \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}\right]^{p-1}\ \text{converges},\qquad (1<p\leqq 2),$$

we state the following theorems.

THEOREM 1. *The system* (1) *is equivalent to* (*that is, has the same solution as*) *one which has an absolutely convergent determinant* Δ *in the sense of von Koch.*

THEOREM 2. *If* $\Delta\neq 0$ *and* $\sum_{i=1}^{\infty}|y_i|^p$ *converges, then the solution is obtainable from the new system by Cramer's rule and is in the same space.*

THEOREM 3. *The inverse of* (1) *is equivalent to a system with an absolutely convergent determinant whose elements satisfy* (3).

THEOREM 4. *If* (1) *is transposed, it becomes equivalent to a system with an absolutely convergent determinant and transforms the space* (x_k), *where* $\sum_k|x_k|^{p/(p-1)}$ *converges,* $(p/(p-1)\geqq 2)$, *into the same space. The transposed system has the properties of Theorems* 2 *and* 3 *with* p *replaced by* $p/(p-1)$ *and* i *and* k *interchanged.*

Theorems 1 and 2 are slight extensions of results obtained by Bóbr,* while 3 and 4 seem to be new.

We first establish the following lemma.

LEMMA. *If, in the determinant* $\Delta=|\delta_{ik}+a_{ik}|$, $(i,k=1,2,\cdots)$, $\sum_{i=1}^{\infty}|a_{ii}|$ *and* $\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}\right]^{p-1}$, $(i\neq k;\ 1<p\leqq 2)$, *converge, then*

$$(4)\qquad \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}|a_{ik}|^{p}\right]^{1/(p-1)},$$

$$(5)\qquad \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}|A_{ik}|^{p/(p-1)}\right]^{p-1},\qquad (i\neq k),$$

$$(6)\qquad \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|A_{ik}|^{p}\right]^{1/(p-1)}$$

converge and $\lim_i A_{ii}=\Delta$, *where* A_{ik} *is the cofactor of* a_{ik}.

* Bóbr, *Eine Verallgemeinerung des v. Kochschen Satzes* $\cdots$, Mathematische Zeitschrift, vol. 10 (1921), pp. 1–11. Bobr's conditions on (1) are that $\sum_{i=1}^{\infty}|a_{ii}|$ and $\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}\right]^{p-1}(i\neq k;\ 1<p\leqq 2)$ converge. The absolute convergence of the determinant is established and the solution is obtained by indirect methods involving the theory of limited linear and bilinear forms.

PROOF: The determinant and its minors converge absolutely and the following relations hold:*

$$(7)\qquad \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|a_{ik}|^{p/(p-1)}\right]^{p-1}=\sigma^{p},\qquad (i\neq k),$$

$$(8)\qquad \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty}|a_{ik}|^{p}\right]^{1/(p-1)}=\tau^{p/(p-1)}\leqq\sigma^{p/(p-1)},$$

$$(9)\qquad \sum_{i_{n+1}=1}^{\infty}\left[\sum_{i_2,\cdots,i_n=1}^{\infty}|i_1,i_2,\cdots,i_n;i_{n+1}|\right]^{p/(p-1)}\leqq\tau^{(n-1)p/(p-1)}\sum_{i_{n+1}=1}^{\infty}|i_1;i_{n+1}|^{p/(p-1)},$$

$$(10)\qquad \sum_{i_1=1}^{\infty}\left[\sum_{i_2,\cdots,i_n=1}^{\infty}|i_1,i_2,\cdots,i_n;i_{n+1}|\right]^{p}\leqq\sigma^{(n-1)p}\sum_{i_1=1}^{\infty}|i_1,i_{n+1}|^{p},$$

where $|i_1, i_2, \cdots, i_n; i_{n+1}| = |a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_{n+1}}|$ if all the subscripts are distinct and is zero otherwise.

By the interchange of a finite number of rows or columns in A_{ik}, its diagonal can be made to differ from that of Δ in that $1+a_{ii}$, and $1+a_{kk}$ are missing and a_{ki} occurs. Then A_{ik} may be expanded as a sum of infinite products of the form

$$(11)\qquad c_{1j_1}c_{2j_2}\cdots c_{mj_m}\prod_{\mu=m+1}^{\infty}c_{\mu\mu},$$

where c_{hj} is the general term of A_{ik}, $(j_1, j_2, \cdots, j_m)$ is a permutation of $(1, 2, \cdots, m)$ and the $c_{\mu\mu}$ are on the diagonal of A_{ik}. If we replace, in each product*, c_{hj} by $\delta_{hj}+a_{hj}$ and expand, we obtain new products in the a_{hj}. Each of these except for sign is a product of a_{ki} or $|k, i_1, \cdots, i_n; i|$ $(n\geqq 1)$ multiplied by a product of circular products of Δ.† If we put $P=\Pi(1+|\pi|)$, where the product is extended over all circular products of Δ, we may write, P being finite,

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* Bôbr, loc. cit.

† See F. Riesz, loc. cit., pp. 33–35.

$$(12)\quad |A_{ik}| \leqq P\Big\{|a_{ki}| + \sum_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n=1}^{\infty} |k, i_1, \cdots, i_n; i|\Big)\Big\},$$

$(i \neq k)$. Using the Hölder inequality and (10), we obtain

$$(13)\quad \sum_{i_1,\cdots,i_n=1}^{\infty} |k, i_1, i_2, \cdots, i_n; i|$$

$$= \sum_{i_1=1}^{\infty} |k; i_1| \sum_{i_2,\cdots,i_n=1}^{\infty} |i_1, i_2, \cdots, i_n; i|$$

$$\leqq \left[\sum_{i_1=1}^{\infty} |k; i_1|^{p/(p-1)}\right]^{(p-1)/p}$$

$$\cdot \left[\sum_{i_1=1}^{\infty}\left\{\sum_{i_2,\cdots,i_n=1}^{\infty} |i_1, i_2, \cdots, i_n; i|\right\}^p\right]^{1/p}$$

$$\leqq \sigma^{n-1}\left[\sum_{i_1=1}^{\infty} |k; i_1|^{p/(p-1)}\right]^{(p-1)/p}\left[\sum_{i_1=1}^{\infty} |i_1; i|^p\right]^{1/p}.$$

Combining this with (12) and making the assumption that $\sigma < 1$, we have

$$(14)\quad |A_{ik}| \leqq P\left\{|a_{ki}| + \frac{1}{1-\sigma}\left[\sum_{i_1=1}^{\infty} |k; i_1|^{p/(p-1)}\right]^{(p-1)/p}\left[\sum_{i_1=1}^{\infty} |i_1; i|^p\right]^{1/p}\right\}$$

$$(k \neq i).$$

Let us consider a fixed k and apply the Minkowski inequality:

$$(15)\quad \left[\sum_{i=1}^{\infty} |A_{ik}|^{p/(p-1)}\right]^{(p-1)/p} \leqq P\Bigg(\sum_{i=1}^{\infty}\left\{|a_{ki}| + \frac{1}{1-\sigma}\left[\sum_{i_1=1}^{\infty} |k; i_1|^{p/(p-1)}\right]^{(p-1)/p}\left[\sum_{i_1=1}^{\infty} |i_1; i|^p\right]^{1/p}\right\}^{p/(p-1)}\Bigg)^{(p-1)/p}$$

$$\leqq P\left\{\sum_{i=1}^{\infty} |a_{ki}|^{p/(p-1)}\right\}^{(p-1)/p} + \frac{P}{1-\sigma}\left[\sum_{i_1=1}^{\infty} |k; i_1|^{p/(p-1)}\right]^{(p-1)/p}\left\{\sum_{i=1}^{\infty}\left[\sum_{i_1=1}^{\infty} |i_1; i|^p\right]^{1/(p-1)}\right\}^{(p-1)/p}.$$

Since $|k; i_1| = |a_{ki_1}|$ or 0 according as $i_1 \neq$ or $= k$, we have, using (8),

$$(16)\quad \left[\sum_{i=1}^{\infty} |A_{ik}|^{p/(p-1)}\right]^{(p-1)/p} \leqq \left(P\frac{1-\sigma+\tau}{1-\sigma}\right)\left\{\sum_{i=1}^{\infty} |a_{ki}|^{p/(p-1)}\right\}^{(p-1)/p}, \qquad (i \neq k).$$

Then raising to the pth power, summing as to k and using (7), we have

$$(17)\quad \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} |A_{ik}|^{p/(p-1)}\right]^{p-1} \leqq \left(P\frac{1-\sigma+\tau}{1-\sigma}\right)^{p}\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} |a_{ki}|^{p/(p-1)}\right]^{p-1} \leqq \left(P\sigma\frac{1-\sigma+\tau}{1-\sigma}\right)^{p}, \qquad (i \neq k).$$

If we set $|k, i_1, \cdots, i_{n-1}, i_n; i| = |i_n; i|\ |k, i_1, \cdots, i_{n-1}; i_n|$ and use (9), we are led to

$$(18)\quad \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty} |A_{ik}|^{p}\right]^{1/(p-1)} \leqq \left(P\tau\frac{1-\tau+\sigma}{1-\tau}\right)^{p/(p-1)} \qquad (k \neq i).$$

Hence we have established (5) and (6) in case $\sigma < 1$. This restriction may be removed as follows. Since

$$\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)}\right]^{p-1}, \qquad (k \neq i),$$

converges, there is an integer m such that

$$(19)\qquad \sum_{i=m+1}^{\infty}\left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)}\right]^{p-1} < \frac{1}{2}, \qquad (k \neq i).$$

There is a number t $(0 < t < 1)$ such that

$$t^p \sum_{i=1}^{m} \left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} \right]^{p-1} < \frac{1}{2}. \tag{20}$$

If we multiply the first m rows of Δ by t, then in the new determinant Δ', $\sigma<1$ and (5) and (6) hold for Δ'. Both determinants and their minors are absolutely convergent. If the A'_{ik} are the cofactors of the elements of Δ', they differ from the A_{ik} in that the first m or $m-1$ rows are multiplied by t. But t being less than 1, we have in any case

$$t^m |A_{ik}| \leqq |A'_{ik}|, \tag{21}$$

and consequently

$$\begin{cases} \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} |A_{ik}|^{p/(p-1)} \right]^{p-1} \\ \qquad \leqq t^{-mp} \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} |A'_{ik}|^{p/(p-1)} \right]^{p-1}, \\ \sum_{i=1}^{\infty} \left[\sum_{k=1}^{\infty} |A_{ik}|^{p} \right]^{1/(p-1)} \\ \qquad \leqq t^{-mp/(p-1)} \sum_{i=1}^{\infty} \left[\sum_{k=1}^{\infty} |A'_{ik}|^{p} \right]^{1/(p-1)}, \end{cases} \qquad (i \neq k), \tag{22}$$

so that (5) and (6) hold in general.

Finally we show that $\lim_i A_{ii}=\Delta$. Since Δ is absolutely convergent, we have $\Delta=\sum_{k=1}^{\infty}(\delta_{ik}+a_{ik})A_{ik}$ for all i. From the Hölder inequality we get

$$|\Delta - (1+a_{ii})A_{ii}| \leqq \sum_{k=1}^{\infty} |a_{ik}A_{ik}|$$

$$\leqq \left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} \right]^{(p-1)/p} \left[\sum_{k=1}^{\infty} |A_{ik}|^{p} \right]^{1/p}, \qquad (k \neq i). \tag{23}$$

Applying the inequality $A^{1/p}B^{(p-1)/p} \leqq (1/p)A + [(p-1)/p]B$, $(A \geqq 0, B \geqq 0, p>1)$, we have

$$(24)\qquad |\Delta - (1+a_{ii})A_{ii}| \leqq (1/p)\left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)}\right]^{p-1} + [(p-1)/p]\left[\sum_{k=1}^{\infty} |A_{ik}|^{p}\right]^{1/(p-1)}, \qquad (k \neq i).$$

Now in consequence of (7), (22), and the fact that $\lim_i a_{ii}=0$, we have

$$(25)\qquad \lim_i A_{ii} = \Delta.$$

We return now to our system (1) under condition (3).

I. It follows from (3) that $\lim_i a_{ii}=0$. Hence there is an i_0 such that for $i>i_0$, $1/2<1/(1+a_{ii})<2$. We put $d_i=1$ for $i\leqq i_0$ and $d_i=1/(1+a_{ii})$ for $i>i_0$. Multiplying the ith equation of (1) by d_i, we obtain

$$(26)\qquad x_i + \sum_{k=1}^{\infty} c_{ik}x_k = z_i, \qquad (\delta_{ik} + c_{ik} = d_i(\delta_{ik} + a_{ik});\ z_i = d_i y_i),$$

which has the same solution as (1). Since the d_i are bounded and the $c_{ii}=0$ for $i>i_0$, $\sum_{i=1}^{\infty}|c_{ii}|$ and $\sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}|c_{ik}|^{p/(p-1)}\right]^{p-1}$ converge. It follows that (26) has an absolutely convergent determinant.

II. We solve (26) by Cramer's rule. Let $\Delta^{(j)}$ be the determinant obtained by replacing the jth column of $\Delta=|\delta_{ik}+c_{ik}|$ by the z_i and consider

$$(27)\qquad \Delta^{(j)} + \sum_{k=1}^{\infty} c_{jk}\Delta^{(k)} = \sum_{i=1}^{\infty} z_i C_{ij} + \sum_{k=1}^{\infty} c_{jk}\sum_{i=1}^{\infty} z_i C_{ik}$$

$$= \sum_{i=1}^{\infty} z_i\left[C_{ij} + \sum_{k=1}^{\infty} c_{jk}C_{ik}\right]$$

$$= \sum_{i=1}^{\infty} z_i \delta_{ij}\Delta = z_j\Delta,$$

where C_{ik} is the cofactor of $\delta_{ik}c_{ik}$. This formal procedure is justified if the double series involved converges absolutely. But, from the Hölder inequality, we have

$$(28)\quad \sum_{k=1}^{\infty} |c_{jk}| \sum_{i=1}^{\infty} |z_i C_{ik}| = \sum_{k=1}^{\infty} |c_{jk}| \left\{ |z_k C_{kk}| + \sum_{i=1}^{\infty} |z_i C_{ik}| \right\}$$

$$(i \neq k \text{ on the right})$$

$$\leqq \sum_{k=1}^{\infty} |c_{jk}| \left\{ |z_k C_{kk}| + \left[\sum_{i=1}^{\infty} |z_i|^p \right]^{1/p} \left[\sum_{i=1}^{\infty} |C_{ik}|^{p/(p-1)} \right]^{(p-1)/p} \right\}$$

$$\leqq C \sum_{k=1}^{\infty} |c_{jk} z_k| + \left[\sum_{i=1}^{\infty} |z_i|^p \right]^{1/p} \sum_{k=1}^{\infty} |c_{jk}| \left[\sum_{i=1}^{\infty} |C_{ik}|^{p/(p-1)} \right]^{(p-1)/p}$$

$$\leqq \left[\sum_{i=1}^{\infty} |z_i|^p \right]^{1/p} \left[\sum_{k=1}^{\infty} |c_{jk}|^{p/(p-1)} \right]^{(p-1)/p} \cdot \left\{ C + \left(\sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} |C_{ik}|^{p/(p-1)} \right]^{p-1} \right)^{1/p} \right\}, \qquad (i \neq k),$$

where C is the least upper bound of the $|C_{kk}|$. The lemma by (5) guarantees the convergence of the double series on the right and hence our series converges absolutely. We have then the solution $x_k = \Delta^{(k)}/\Delta$. That the solution is of the required type results as follows, since $\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} |\Delta^{(k)}/\Delta|^p$:

$$(29)\quad |\Delta^{(k)}|^p \leqq \left[\sum_{i=1}^{\infty} |z_i C_{ik}| \right]^p = \left[|z_k C_{kk}| + \sum_{i=1}^{\infty} |z_i C_{ik}| \right]^p, \quad (i \neq k \text{ on the right}),$$

$$(30)\quad \left[\sum_{k=1}^{\infty} |\Delta^{(k)}|^p \right]^{1/p} \leqq \left[\sum_{k=1}^{\infty} |z_k C_{kk}|^p \right]^{1/p} + \left[\sum_{k=1}^{\infty} \left\{ \sum_{i=1}^{\infty} |z_i C_{ik}| \right\}^p \right]^{1/p}$$

$$\leqq \left[\sum_{k=1}^{\infty} |z_k|^p \right]^{1/p} \left(C + \left\{ \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} |C_{ik}|^{p/(p-1)} \right]^{p-1} \right\}^{1/p} \right),$$

($i \neq k$ on the right). The lemma again yields the convergence of the double series on the right.

III. In order to obtain the inverse of (1), we consider the inverse of (26)

$$z_k + \sum_{i=1}^{\infty} \gamma_{ik} z_i = x_k, \quad (\delta_{ik} + \gamma_{ik} = C_{ik}/\Delta). \tag{31}$$

The inverse of (1) then is

$$\frac{y_k}{d_k} + \sum_{i=1}^{\infty} \frac{\gamma_{ik}}{d_i} y_i = x_k. \tag{32}$$

Here $\lim_i \gamma_{ii} = 0$ and $\sum_{k=1}^{\infty} [\sum_{i=1}^{\infty} |\gamma_{ik}|^{p/(p-1)}]^{p-1}$ $(i \neq k)$ converges. A set of constants $\delta_k (1/2 < \delta_k < 2)$ may be determined as in I so that, in the system

$$y_k + \sum_{i=1}^{\infty} \alpha_{ik} y_i = u_k, \tag{33}$$

we have

$$\delta_{ik} + \alpha_{ik} = (\delta_{ik} + \gamma_{ik}) \frac{d_k \delta_k}{d_i}; \quad u_k = d_k \delta_k x_k, \tag{34}$$

$$\alpha_{kk} = 0 \ (k > k_0); \quad |\alpha_{ik}| \leqq \frac{8}{|\Delta|} |C_{ik}|, \quad (i \neq k). \tag{35}$$

From (35) it follows that $\sum_{i=1}^{\infty} |\alpha_{ii}|$ and $\sum_{k=1}^{\infty} [\sum_{i=1}^{\infty} |\alpha_{ik}|^{p/(p-1)}]^{p-1}$ converge. Hence the system (33), equivalent to the inverse of (1), has an absolutely convergent determinant.

IV. If (1) is transposed, it becomes

$$x_k + \sum_{i=1}^{\infty} a_{ik} x_i = y_k. \tag{36}$$

From (3) it follows that $1/(p-1) \geqq 1$ and we have, applying the Minkowski inequality if the equality here does not hold,

$$\left\{ \sum_{k=1}^{\infty} \left[\sum_{i=1}^{\infty} |a_{ik}|^p \right]^{1/(p-1)} \right\}^{p-1} \leqq \sum_{i=1}^{\infty} \left[\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} \right]^{p-1}. \tag{37}$$

Therefore $\sum_{i=1}^{\infty} |a_{ik}|^p$ converges for every k and (36) is a transformation on the space (x_i), where $\sum_{i=1}^{\infty} |x_i|^{p/(p-1)}$ converges and $p/(p-1) \geqq 2$. For any such (x_i), we have

$$(38)\quad \left[\sum_k |y_k|^{p/(p-1)}\right]^{(p-1)/p} \leqq \left[\sum_{i=1}^{\infty} |x_i|^{p/(p-1)}\right]^{(p-1)/p} \cdot \left[1 + \left\{\sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} |a_{ik}|^p\right]^{1/(p-1)}\right\}^{(p-1)/p}\right],$$

and hence the (y_k) is in the same space. On the other hand, given (y_k) in this space, we find an equivalent system (26′) as in I having an absolutely convergent determinant and obtain the solution from (26′) by Cramer's rule as in II, (6) of the lemma being used. The discussion of the inverse of the transposed system follows that of III.

PRINCETON UNIVERSITY

EXAMPLES OF SURFACE TRANSFORMATIONS

BY P. A. SMITH

The purpose of this note is to illustrate by examples certain points in the structure analysis of surface transformations, and at the same time to point out certain unsolved problems which appear to be fundamental in this connection.

Every (1-1) continuous transformation of a surface S into itself admits* a closed invariant set of *central points* which possesses certain properties of regional recurrence and which is determined essentially as follows: the points of S move under indefinite iteration of T and its inverse T_{-1}, toward a certain closed invariant set M^1 which in general is a proper subset of S; M^1 contains a proper subset M^2 related to M^1 as M^1 is to S. Continuing thus, we finally arrive at a first set M^r such that

* The definitions and theorems used implicitly in this note are to be found in detail in Birkhoff and Smith, *Structure analysis of surface transformations*, Journal de Mathématiques, vol. 7 (1928). With regard to regular regions, see also P. A. Smith, *Regular components of surface transformations*, to appear shortly in the American Journal of Mathematics.

$M^{r+i}=M^r$ $(i=1, 2, \cdots)$ and M^r is the set in question. The number r is a sort of measure of the structural complexity of T, and may perhaps be a transfinite ordinal. The fact is, however, that as yet no examples are known for which $r>2$. Examples with $r=1$ are simple to construct; a sphere, for example, may be rotated through 180°. Then $M^1=S$ and hence $M^i=S$, $i>1$, and $M^r=S$, $r=1$. Transformations with $r=2$ can also be effectively constructed; to show this will be the purpose of our first example. It would be of considerable value for structure analysis to know whether or not r can be greater than 2.

For the set M^1 we may take the totality of *non-wandering* points of S,—they are defined by the property that an arbitrarily small neighborhood σ of any of them is intersected by images of σ under sufficiently great powers of T and T_{-1}. Or, we may start with N^1 for our initial set, N^1 being the totality N of α- and ω-limit points of T, together with the ordinary limit points of N. Here we may ask if N is not itself closed, and our second example is constructed to show that such need not be the case. Each point of N^1 is non-wandering and hence there can exist non-wandering points which are not α- or ω-limit points. It would be of considerable interest to go a step farther and learn whether or not there can exist non-wandering points not contained in N^1. A closely related unsolved problem is the following: the sequence N^1, N^2, $\cdots$ leads eventually to a final set N^s which is identical with M^r; it is known that s is not greater than r,—must $s=r$?

The set $S-M^1$, if it is not null, may contain certain *regular regions*, that is, regions of uniform approach toward M^1 with respect to indefinite iteration of T. The question arises as to whether such regions *must* exist. We shall describe briefly a transformation in which the open set $S-M^1$ contains no regular regions.

The Transformations. Let S be a sphere and $\theta=$const., $\phi=$const. $(-\pi/2\leqq\theta\leqq\pi/2, 0\leqq\phi\leqq 2\pi)$ be its parallels and meridians respectively. We shall first describe three preliminary transformations of S.

A. Let t be a (1-1) continuous transformation of the equator $(\theta=0)$ into itself, leaving invariant the point O $(\theta=0, \phi=0)$ and increasing the ϕ-coordinate of every other point by an

amount less than $\pi/2$. If t is represented by $\phi_1=\phi+f(\phi)$, the first transformation A of S will be defined by

$$\theta_1 = \theta, \quad \phi_1 = \phi + f(\phi).$$

Thus A leaves invariant the poles $\theta=\pm\pi/2$, and the points of the meridian $\phi=0$, and increases the ϕ of all other points by amounts always less than $\pi/2$.

B. The transformation B is given by

$$\theta_1 = \theta - \lambda \sin 2\theta, \qquad \phi_1 = \phi,$$

where the positive constant λ is chosen, as it can be, so small that B shall be (1-1). The only invariant points here are the poles and the points of the equator. The general motion of all other points on iteration of B is toward the equator. In fact if $\cdots C_{-1}, C_0, C_1, \cdots$ represent the successive images of the parallel C_0 in the upper (lower) hemisphere, the sequence $C_0, C_1, \cdots$ has the equator for its only limit circle, and the sequence $C_0, C_{-1}, \cdots$ closes down on the pole $\theta=\pi/2, (\theta=-\pi/2)$.

C. This transformation is represented by

$$\theta_1 = \theta, \qquad \phi_1 = \phi + g(\theta),$$

where $g(\theta)$ is defined as follows: assume C_0 of the preceding paragraph to lie in the upper hemisphere, and let C_n be given by $\theta=\theta_n, (n=0, \pm 1, \cdots)$. Then let $g(\theta_n)=\pi(|n|+1)^{-1}$, $g(0)=g(\pi/2)=0$; on each interval (θ_n, θ_{n+1}), $g(\theta)$ is to vary linearly between the values taken at the end points. This defines $g(\theta)$ for $0\leqq\theta\leqq\pi/2$ and the definition is now completed by putting $g(-\theta)=g(\theta)$.

Consider now the product transformation $T=ABC$; *we shall show that for* T, $r=2$. We note first that T may be represented in the form

$$\theta_1 = \theta - \lambda \sin 2\theta, \qquad \phi_1 = F(\theta,\phi).$$

Hence, as for B, motion in the two hemispheres is uniformly toward the equator on iteration of T, and toward their respective poles (which are invariant points) on iteration of T_{-1}. Hence the set M^1 is contained in the set P^1+P^2+equator, P^1 and P^2 being the poles. The equator transforms under T just as it does under A, since its points are invariant under B

and C. Hence its points move toward the invariant point O on iteration of T and T_{-1}. Hence M^2 contains only the points O, P^1, P^2. Thus $M^r = M^2$, and $r \leqq 2$. To show that $r = 2$, we must show that M^2 is a proper subset of M^1. We shall show in fact that all points of the equator are in M^1.

Let σ be a small neighborhood of an arbitrary point P of the equator. For m positive and sufficiently large, σ will be intersected by C_m; let Q be a point of intersection. Now the ϕ-coordinate of Q receives non-negative increments from A and B; hence its increment from T is at least as great as that due to C, namely $\pi(1+m)^{-1}$. Hence the ϕ of Q is increased by an amount $\geqq \sum_{i=0}^{k-1} \pi(m+1+i)^{-1}$ on k applications of T, and is therefore increased without bound on infinite iteration. Thus Q describes, so to speak, infinitely many circuits about the sphere in the neighborhood of the equator, whereas P converges directly to O. Hence the successive images of σ tend to wind themselves around the sphere, and are at the same time drawn constantly closer to the equator. Clearly σ_n must intersect σ for large positive values of n, and P is therefore non-wandering, hence in M^1. This completes the proof that $r = 2$.

Consider next the transformation AC; we shall show that the set N defined above is not closed. We remark first that AC transforms into itself every circle $\theta = c$ according to the relation

$$\phi_1 = \phi + f(\phi) + g(c) = \lambda_c(\phi).$$

Since $f(\phi) \geqq 0$ for all ϕ, and $g(c) > 0$ except for $c = 0, \pm\pi/2$, the only invariant points are the poles and O. The function $\lambda_c(\phi)$ undergoes continuous modification as c varies continuously and hence the rotation number* $\rho(c)$ associated with $\lambda_c(\phi)$ is a continuous function of c as one can readily verify. Since O is invariant, $\rho(0) = 0$ and hence an arbitrarily small non-zero value c' of c can be chosen such that $\rho(c')$ is commensurable with π. Because of a property of rotation numbers, there exist on the circle $\theta = c$ (which is arbitrarily near the equator) a number of periodic points. Let one of them be U. Since U can not be invariant, it is at least of order 2, and iterates periodically

* For a definition of rotation numbers and an account of their properties, see Birkhoff, *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1920), p. 87.

through a finite group of two or more periodic points on repeated application of AC. Hence the ϕ-coordinate of U increases without bound. But the successive increments of ϕ are $<\pi/2$. This is due to the definition of A and the fact that near the equator the influence of C is minute. It follows that there exists at least one periodic point on every quarter-circumference of $\theta=c'$. Clearly the totality of periodic points must therefore have at least four limit points on the equator. Since each periodic point is an α- and ω-limit point, and since the only α- or ω-limit point on the equator is 0, we have at least three points which are not α- or ω-limit points, but are limit points of such points. *Hence N is not closed.*

In our final example* we shall let S be represented by the x y plane with a single point at infinity. Each point of S not contained between the two lines $y=O$ and $y=1$ is to be invariant. The only further invariant points are those of the set $(\pm p/q,\ 1/q)$, where p and q are positive integers and $q>1$. All the remaining points are to move vertically downward $(x_1=x,\ y_1<y)$ on iteration of T, and hence vertically upward on iteration of T_{-1}. Clearly the set M^1 consists of the invariant points only.

Let σ be a small connected region in $S-M^1$. We can choose in σ a point $P(a,\ b)$ where $a=m/n,\ b>1/n$, and a point $P'(a',\ b')$ where a' is irrational. Under iteration of T, all points of σ move vertically downward. The sequence $P,\ P_1,\ P_2,\ \cdots$ converges to $V(m/n,\ 1/n)$, while the sequence $P',\ P_1',\ P_2',\ \cdots$ converges to $(a',\ 0)$. Now V is an isolated point of M^1 and is at a non-zero distance δ from the remaining points of M^1. Hence no image of σ under positive powers of T can lie entirely within a distance ϵ of M^1, if we take $\epsilon<\delta/2$. Thus *there are no regions of uniform approach toward* M^1 and all points of $S-M^1$ are ω-irregular.

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* Certain details of this construction will be omitted; they are not difficult, however, and have been carefully verified.

A SOLUTION OF A SIMPLE FUNCTIONAL EQUATION AS A BASIS FOR READILY OBTAINING CERTAIN FUNDAMENTAL FORMULAS IN THE THEORY OF ELLIPTIC FUNCTIONS*

BY G. W. STARCHER

The purpose of the present note is to show, by considering certain solutions of simple functional equations, how one may obtain expressions for each of the four theta functions, both as infinite products and as Fourier series, and thus have a basis for the theory of elliptic functions. The principal novelty of the note lies in the fact that the two fundamental expressions for the theta functions are obtained at once from considering simple functional equations; moreover, an important relation between theta functions of zero argument is obtained with less effort than is usually given to it. Throughout this discussion we assume q to be such that $|q|<1$.

The functional equation

$$f(x) = (1 + qx)f(qx) \tag{1}$$

has a unique solution analytic in the neighborhood of the point zero, having there the value 1; this solution is expressible in each of the two forms

$$\prod_{v=1}^{\infty}(1 + q^v x) = 1 + \sum_{v=1}^{\infty} \frac{q^{(1/2)v(v+1)}}{(1-q)(1-q^2)\cdots(1-q^v)} x^v. \tag{2}$$

From (1), we see that $g(x)=1/f(-x)$ satisfies the equation

$$g(x) = \frac{1}{1-qx} g(qx). \tag{3}$$

This equation has a unique solution analytic in the neighborhood of the point zero, having there the value 1, which is expressible as an infinite product by comparison with the expression for $f(x)$ as an infinite product. If such a solution of (3) may be written in the form

* Presented to the Society, April 18, 1930.

$$g(x) = 1 + \sum_{v=1}^{\infty} \frac{c_v x^v}{(1-qx)(1-q^2x)\cdots(1-q^vx)},$$

then we have

$$\frac{1}{1-qx}g(qx) = \frac{1}{1-qx} + \sum_{v=1}^{\infty} \frac{c_v q^v x^v}{(1-qx)(1-q^2x)\cdots(1-q^vx)} \cdot \frac{1}{(1-q^{v+1}x)}$$

$$= 1 + \frac{qx}{1-qx} + \sum_{v=1}^{\infty} \frac{c_v q^v x^v}{(1-qx)(1-q^2x)\cdots(1-q^vx)} + \sum_{v=1}^{\infty} \frac{c_v q^{2v+1} x^{v+1}}{(1-qx)(1-q^2x)\cdots(1-q^{v+1}x)}.$$

Equating coefficients of like terms and computing the successive c's we have finally the solution

$$g(x) = 1 + \sum_{v=1}^{\infty} \frac{q^v}{(1-q)(1-q^2)\cdots(1-q^v)} \frac{x^v}{(1-qx)(1-q^2x)\cdots(1-q^vx)}.$$

For $x = 1$, we have

$$(4) \qquad \frac{1}{\prod_{v=1}^{\infty}(1-q^v)} = 1 + \sum_{v=1}^{\infty} \frac{q^v}{(1-q)^2(1-q^2)^2\cdots(1-q^v)^2}.$$

Consider the functional equation

$$(5) \qquad f(x) = xf(qx).$$

If there exists a solution of this equation, not identically zero, which vanishes when $x = -1$, having no other zeros x_1 such that $|q| < |x_1| \leqq 1$, then all the zeros of such a solution are at the points $x = -q^v$, where $v = \cdots, -2, -1, 0, 1, 2, \cdots$. We may construct a function having precisely these zeros, viz.,

$$(6) \qquad f_1(x) = \prod_{v=1}^{\infty}(1 + q^{v-1}x)(1 + q^v x^{-1}).$$

This infinite product obviously converges and defines a single-valued analytic function for every x except $x = 0$ and $x = \infty$. Substituting $f_1(x)$ for $f(x)$ in (5) we see that $f_1(x)$ satisfies (5).

Any solution of (5) which is single-valued and analytic throughout the plane except at $x=0$ and $x=\infty$, may be represented by a Laurent series which may be written,

$$f(x) = \sum_{-\infty}^{\infty} c_v x^v. \tag{7}$$

Substituting from (7) into (5) we have

$$\sum_{-\infty}^{\infty} c_v x^v = \sum_{-\infty}^{\infty} c_v q^v x^{v+1}.$$

Computing the successive c's in terms of c_0 and substituting these values into (7) we have

$$f(x) = c_0 \sum_{-\infty}^{\infty} q^{(1/2)v(v-1)} x^v. \tag{8}$$

This series converges and defines a function which is single-valued and analytic throughout the plane except at $x=0$ and $x=\infty$. Hence a solution of (5) which is single-valued and analytic throughout the plane except at $x=0$ and $x=\infty$ is uniquely determined by (5) except for a multiplier c_0 which is independent of x. For some particular c_0 the series in (8) must define the same solution as (6). To find the required value of c_0, we observe that c_0 is the term in the series that is independent of x, and determine the term independent of x in the expansion of the infinite product (6). From (2) when $x=q^{-1}x$ we have

$$\prod_{v=1}^{\infty}(1+q^{v-1}x) = 1 + \sum_{v=1}^{\infty} \frac{q^{(1/2)v(v-1)}}{(1-q)(1-q^2)\cdots(1-q^v)} x^v,$$

and when $x=x^{-1}$

$$\prod_{v=1}^{\infty}(1+q^v x^{-1}) = 1 + \sum_{v=1}^{\infty} \frac{q^{(1/2)v(v+1)}}{(1-q)(1-q^2)\cdots(1-q^v)} x^{-v}.$$

In the product of the two expressions on the right the sum of the products of terms involving x^v in the first by the terms involving x^{-v} in the second, together with the product of the two constant terms, gives precisely the term independent of x in the expansion of (6). This is clearly just the second mem-

ber of the identity (4). Writing the first member of (4) for c_0 in (8) and equating to the product in (6) we obtain

$$(9)\qquad \prod_{r=1}^{\infty}(1+q^{r-1}x)(1+q^{r}x^{-1})(1-q^{r}) = \sum_{-\infty}^{\infty}q^{(1/2)r(r-1)}x^{r}.$$

Replacing x by $-x$ we also have

$$(10)\qquad \prod_{r=1}^{\infty}(1-q^{r-1}x)(1-q^{r}x^{-1})(1-q^{r}) = \sum_{-\infty}^{\infty}(-1)^{r}q^{(1/2)r(r-1)}x^{r}.$$

Replacing q by q^2 and x by qe^{2iz} and using the customary notation $\vartheta_3(z)$, $\vartheta_4(z)$ for the theta functions, we have from (9) and (10)

$$(11)\qquad \prod_{r=1}^{\infty}(1+q^{2r-1}e^{2iz})(1+q^{2r-1}e^{-2iz})(1-q^{2r})$$

$$= \sum_{-\infty}^{\infty}q^{r^2}e^{2riz} = \vartheta_3(z),$$

$$(12)\qquad \prod_{r=1}^{\infty}(1-q^{2r-1}e^{2iz})(1-q^{2r-1}e^{-2iz})(1-q^{2r})$$

$$= \sum_{-\infty}^{\infty}(-1)^{r}q^{r^2}e^{2riz} = \vartheta_4(z).$$

Again replacing q by q^2 and x by q^2e^{2iz}, we have from (9) after multiplying by $q^{1/4}e^{iz}$

$$(13)\qquad q^{1/4}e^{iz}\prod_{r=1}^{\infty}(1+q^{2r}e^{2iz})(1+q^{2r-2}e^{-2iz})(1-q^{2r})$$

$$= \sum_{-\infty}^{\infty}q^{(r+1/2)^2}e^{(2r+1)iz} = \vartheta_2(z).$$

Similarly from (10) after multiplying by $-iq^{1/4}e^{iz}$

$$(14)\qquad -iq^{1/4}e^{iz}\prod_{r=1}^{\infty}(1-q^{2r}e^{2iz})(1-q^{2r-2}e^{-2iz})(1-q^{2r})$$

$$= \sum_{-\infty}^{\infty}(-1)^{r}q^{(r+1/2)^2}e^{(2r+1)iz} = \vartheta_1(z).$$

Thus we have each of the four theta functions expressed as an infinite product and as an infinite series, the latter being equivalent to the Fourier series for the theta functions. From these it is easy to establish the quasi-periodic character of the theta functions, together with other of their fundamental properties. By considering suitable quotients of the theta functions we obtain elliptic functions, and hence much of the theory of elliptic functions may be obtained readily.

We may easily obtain a fundamental relation between the theta functions of zero argument as follows.

From (14) we may write

$$\vartheta_1(z) = -iq^{1/4}e^{iz}(1-e^{-2iz})\prod_{v=1}^{\infty}(1-q^{2v}e^{2iz})(1-q^{2v}e^{-2iz})(1-q^{2v})$$

$$= 2q^{1/4}\sin z\prod_{v=1}^{\infty}(1-q^{2v}e^{2iz})(1-q^{2v}e^{-2iz})(1-q^{2v}).$$

Differentiating with respect to z, then allowing z to approach zero, we have

$$\vartheta_1'(0) = 2q^{1/4}\prod_{v=1}^{\infty}(1-q^{2v})^3$$

$$= 2q^{1/4}\prod_{v=1}^{\infty}(1-q^{2v})(1+q^v)^2(1-q^v)^2$$

$$= 2q^{1/4}\prod_{v=1}^{\infty}(1-q^{2v})(1+q^{2v})^2(1+q^{2v-1})^2(1-q^{2v})^2(1-q^{2v-1})^2$$

$$= 2q^{1/4}\prod_{v=1}^{\infty}(1+q^{2v})^2(1-q^{2v})\cdot\prod_{v=1}^{\infty}(1+q^{2v-1})^2(1-q^{2v})$$

$$\cdot\prod_{v=1}^{\infty}(1-q^{2v-1})^2(1-q^{2v})$$

$$= {}_2(0)\,{}_3(0)\vartheta_4(0),$$

the last member being obtained by comparison with (11), (12), and (13).

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ON CANONICAL FORMS OF DIFFERENTIAL EQUATIONS*

BY V. G. GROVE

1. *Introduction.* The projective differential geometry of surfaces has been greatly enriched by the introduction of Fubini's projective normal into the theory. This projective normal however fails to exist for ruled surfaces. In this paper we propose a class of congruences covariantly related to the surface, each congruence being suitable as a substitute for Fubini's projective normal congruence. As an application we give an interesting class of congruences covariantly related to a scroll surface and conjugate to the surface.

If the asymptotic parameters are parametric, the four homogeneous coordinates $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ of a general point on the surface S_y are solutions of differential equations in the Wilczynski semicanonical form:†

$$(1) \qquad \begin{cases} y_{uu} + 2ay_u + 2by_v + cy = 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y = 0. \end{cases}$$

One of the integrability conditions of system (1) is

$$(2) \qquad a_v = b_u'.$$

If the transformations

$$(3) \qquad y = \lambda \bar{y}$$

and

$$(4) \qquad \bar{u} = U(u), \quad \bar{v} = V(v)$$

are performed on system (1), the new coefficients are respectively

$$(5) \qquad \begin{cases} \bar{a} = a + \dfrac{\lambda_u}{\lambda}, \quad \bar{b} = b, & \bar{c} = \dfrac{1}{\lambda}(\lambda_{uu} + 2a\lambda_u + 2b\lambda_v + c\lambda), \\ \bar{a}' = a', & \bar{b}' = b' + \dfrac{\lambda_v}{\lambda}, \quad \bar{c}' = \dfrac{1}{\lambda}(\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v + c'\lambda), \end{cases}$$

* Presented to the Society, April 19, 1930.

† E. J. Wilczynski, *First memoir*, Transactions of this Society, vol. 8 (1907), pp. 233–260.

and

$$(6)\qquad \begin{cases} \bar{a} = \dfrac{1}{U_u}\left(a + \dfrac{1}{2}\dfrac{U_{uu}}{U_u}\right), \quad \bar{b} = \dfrac{V_v}{U_u^2}b, \quad \bar{c} = \dfrac{c}{U_u^2}, \\ a' = \dfrac{U_u}{V_v^2}a', \quad \bar{b}' = \dfrac{1}{V_v}\left(b' + \dfrac{1}{2}\dfrac{V_{vv}}{V_v}\right), \quad \bar{c}' = \dfrac{c'}{V_v^2}. \end{cases}$$

2. *The Canonical Forms.* Consider an invariant R of the surface with the following properties:

(a) $R \neq 0$, (b) the transform $\overline{R}$ of R by (4) is $\overline{R} = R/(U_u V_v)$.

Condition (2) implies the existence of a function p such that

$$(7)\qquad p_u = a + h\frac{R_u}{R}, \quad p_v = b' + h\frac{R_v}{R},$$

where h is a constant. The transforms $\bar{p}_u$ and $\bar{p}_v$ of p_u and $\bar{p}_v$ by means of (3) are

$$(8)\qquad \bar{p}_u = p_u + \frac{\lambda_u}{\lambda}, \quad \bar{p}_v = p_v + \frac{\lambda_v}{\lambda}.$$

If we examine the transforms of p_u and p_v under (4), we observe that these functions will be invariant under that transformation if and only if $h = 1/2$. Hence *the points r and s defined by the formulas*

$$(9)\qquad r = y_u + p_u y, \qquad s = y_v + p_v y,$$

$$(10)\qquad p_u = a + \frac{1}{2}\frac{R_u}{R}, \qquad p_v = b' + \frac{1}{2}\frac{R_v}{R},$$

are covariant points. Transformation (3) for $\lambda = e^{-p}$ transforms the expressions r and s into y_u and y_v respectively. With this choice of λ, system (1) assumes the following canonical form:

$$(11)\qquad y_{uu} = \theta_u y_u + \beta y_v + Py, \quad y_{vv} = \gamma y_u + \theta_v y_v + Qy,$$

wherein

$$\theta = \log R, \quad \beta = -2b, \quad \gamma = -2a',$$

$$(12)\qquad \begin{aligned} P &= -\Phi = -[c - a_u - a^2 - 2bb' - b\theta_v - \tfrac{1}{2}\theta_{uu} + \tfrac{1}{4}\theta_u^2], \\ Q &= -\Psi = -[c' - b_v - b'^2 - 2aa' - a'\theta_u - \tfrac{1}{2}\theta_{vv} + \tfrac{1}{4}\theta_v^2]. \end{aligned}$$

The quadratic form

$$(13)\qquad 2R\,du\,dv$$

is absolutely invariant under (3) and (4). If we compute the second covariant derivatives of y with respect to the form (13), we find that the four functions y satisfy the equations

$$y_{11} = \beta y_2 + Py, \quad y_{22} = \gamma y_1 + Qy, \tag{14}$$

wherein $y_1 = y_u$, $y_2 = y_v$.

The coordinates y, which satisfy (11) or (14), are similar to Fubini's normal coordinates. In fact if $R = a'b$ the canonical form (14) becomes Fubini's canonical form and the invariants Φ and Ψ become the invariants ϕ and ψ derived by Lane.*

3. *Geometric Considerations.* We shall call the line joining y to the covariant point y_{uv} the *R conjugate line*, and the line joining the covariant points y_u, y_v the *R harmonic line*. The congruences generated by these lines are respectively conjugate† and harmonic to the surface.

A geometric characterization of the R conjugate line of course depends upon the particular invariant R. In terms of this characterization, geometric interpretations of the covariants y_u, y_v, and y_{uv} and of the vanishing of the invariants P and Q parallel the work of Lane in the paper just cited.

The developables of the congruence of R conjugate lines intersect S_y in the integral curves of the differential equation

$$(\Phi + 2b_v + 2b\theta_v)du^2 - (\Psi + 2a_u' + 2a'\theta_u)dv^2 = 0. \tag{15}$$

The developables of the congruence of R harmonic lines correspond to the curves defined by the equation

$$\Phi du^2 - \Psi dv^2 = 0. \tag{16}$$

Referred to the tetrahedron y, y_u, y_v, y_{uv} of system (11) the equation of the quadric of Lie is

$$x_1x_4 - x_2x_3 + (2a'b + \tfrac{1}{2}\theta_{uv})x_4^2 = 0. \tag{17}$$

The R conjugate line intersects this quadric in y and in the point defined by

* E. P. Lane, *Wilczynski's and Fubini's canonical systems of differential equations*, this Bulletin, vol. 32 (1926), p. 368.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, Transactions of this Society, vol. 20 (1919), p. 99.

$$y_{uv} - (2a'b + \tfrac{1}{2}\theta_{uv})y. \tag{18}$$

4. *Applications to Scroll Surfaces.* Suppose now that the surface S_y is a scroll. Let $v=$const. be the rulings. It follows therefore that $b=0$, $a' \neq 0$. From (15) and (16) we observe that, if we take

$$R = \frac{\nu(v)}{a'} \tag{19}$$

where $\nu(v)$ is a function of v alone, the curves corresponding to the developables of the R conjugate and R harmonic congruences coincide. In order to satisfy property (b) of R, the function $\nu(v)$ must be transformed according to the formula

$$\bar{\nu}(v) = \nu(v)/V_v^3. \tag{20}$$

If we let $z=y_u$, we find readily that the four pairs of functions (y, z) are solutions of differential equations of the form

$$y_{vv} + p_{11}y_v + p_{12}z_v + q_{11}y + q_{12}z = 0,$$
$$z_{vv} + p_{21}y_v + p_{22}z_v + q_{21}y + q_{22}z = 0,$$

wherein

$$\begin{aligned} &p_{11} = 2b', \quad p_{12} = 0, \quad q_{11} = c', \qquad\qquad q_{12} = 2a', \\ &p_{21} = 2b_u', \; p_{22} = 2b', \; q_{21} = c_u' - 2a'c, \; q_{22} = c' + 2a_u' - 4aa'. \end{aligned} \tag{21}$$

Denote by θ_4 the invariant* whose vanishing implies that the ruled surface has the two branches of its flecnode curve coincident. From (21) we find that θ_4 is expressible in terms of the coefficients of system (1) as follows:

$$\theta_4 = 64[(a_u' - 2aa')^2 - 2a'(b_{uv}' - c_u' + 2a'c + 2b'b_u')]. \tag{22}$$

If we use the integrability conditions† of system (1), it is easily verified that $(\theta_4)_u=0$. Hence θ_4 is a function of v alone. Moreover from (6) we find that the transform $\bar{\theta}_4$ of θ_4 by (4) is defined by the formula

$$\bar{\theta}_4 = \theta_4/V_v^4.$$

* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, B. G. Teubner, 1906, p. 104.

† Wilczynski, *First memoir*, p. 245.

We observe that, if the ruled surface has the two branches of its flecnode curve distinct, we may take the function $\nu(v)$ of (19) to be

$$\nu(v) = \theta_4^{3/4}.$$

The R harmonic line, the scroll directrix of the first kind, and the generator intersect in the point y_u for the choice (19) of R. The R conjugate line lies in the plane determined by the scroll directrix of the second kind and the generator.

For the choice (19) of R, and from (18), we see that *the point y_{uv} lies on the osculating quadric for every choice of $\nu(v)$ if and only if*

$$(\log a')_{uv} = 0,$$

that is, if and only if the curved asymptotics on S_y belong to linear complexes. Moreover the point y_u describes a curved asymptotic on S_y as y describes the curve $u = const.$*

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* C. T. Sullivan, *Properties of ruled surfaces whose asymptotic curves belong to linear complexes*, Transactions of this Society, vol. 15 (1914), p. 190.

THEOREMS ON INVERTED AND ROTATED CONGRUENCES*

BY MALCOLM FOSTER

1. *Introduction.* Let l be any line of a rectilinear congruence and M that point on the unit sphere S at which the normal is parallel to l. We refer the sphere to any isothermal system and take the linear element in the form

$$ds^2 = e^{2\lambda}(du^2 + dv^2). \tag{1}$$

At M we consider the moving trihedral of S whose x-axis is chosen tangent to the curves $v=$const., and let (a, b) be the coördinates of the point in which l meets the xy-plane. The conditions that a congruence l be of a particular type are conditions upon the functions a and b.†

For the sphere, $F=D'=0$, and hence

$$\xi_1 = \eta = p = q_1 = 0. \tag{2}$$

Also $E=\mathcal{E}$, $G=\mathcal{G}$, and $\rho_1=\rho_2=-1$;‡ consequently $D=-E$, $D''=-G=-E$. Hence we readily find§

$$\xi = \eta_1 = q = -p_1 = e^{\lambda},\ r = -\frac{\partial\lambda}{\partial v},\quad r_1 = \frac{\partial\lambda}{\partial u}. \tag{3}$$

When the functions in (3) are substituted in the six fundamental relations which are the equivalent of the Gauss and Codazzi equations,‖ all except the equation

$$\frac{\partial^2\lambda}{\partial u^2} + \frac{\partial^2\lambda}{\partial v^2} = -e^{2\lambda}\P \tag{4}$$

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* Presented to the Society, April 18, 1930.

† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ The positive direction of the normal is chosen outward.

§ Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 174.

‖ Eisenhart, p. 168 and p. 170.

¶ Foster, loc. cit. See p. 160 for the solution of this equation.

are identically satisfied. Any solution of (4) determines a particular isothermal system on the sphere S and two adjoint minimal surfaces S_1 and S_2, such that the isothermal system on S is the representation of the lines of curvature on S_1 and of the asymptotic lines on S_2.*

We shall also associate each line of a congruence with those points on the surfaces S_1 and S_2 at which the normals have the same direction. When S_1 and S_2 are referred to their lines of curvature and asymptotic lines respectively, and the linear element of each is taken in the form

$$ds^2 = e^{-2\lambda}(du^2 + dv^2), \tag{5}$$

we readily find that the fundamental quantities for S_1 and S_2 are as follows.†

(6)

For S_1	For S_2
$\xi_1 = \eta = p = q_1 = 0,$	$\xi_1 = \eta = p_1 = q = 0,$
$\xi = \eta_1 = e^{-\lambda},\ p_1 = q = -e^{\lambda},$	$\xi = \eta_1 = e^{-\lambda},\ p = -q_1 = e^{\lambda},$
$r = \dfrac{\partial\lambda}{\partial v},\qquad r_1 = -\dfrac{\partial\lambda}{\partial u}.$	$r = \dfrac{\partial\lambda}{\partial v},\qquad r_1 = -\dfrac{\partial\lambda}{\partial u}.$

2. *The Inversion of a Congruence.* Consider any line l of a congruence and the moving trihedral at the associated point M of S. Relative to a circle C lying in the tangent plane with center at M and of constant radius k, let us invert the point (a, b) in which l meets the tangent plane. The lines l_1 drawn through the inverted points

$$\left(a_1 = \frac{ka}{a^2 + b^2},\quad b_1 = \frac{kb}{a^2 + b^2}\right), \tag{7}$$

parallel to the normals at the associated points M will constitute a congruence which we call the inverse of the congruence formed by the lines l.

Let us suppose that the congruence l is isotropic. Then

$$a = e^{\lambda}\frac{\partial Q}{\partial u},\ b = -e^{\lambda}\frac{\partial Q}{\partial v},\quad \frac{\partial^2 Q}{\partial u^2} + \frac{\partial^2 Q}{\partial v^2} = 0.\ddagger \tag{8}$$

* Eisenhart, p. 252.
† Foster, loc. cit., p. 173.
‡ Foster, loc. cit., p. 173.

We ask: Under what conditions will the congruence l_1 be normal? The condition that a congruence referred to S be normal is*

$$a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad b = e^{-\lambda}\frac{\partial P}{\partial v}, \tag{9}$$

where $P(u, v)$ is an arbitrary function. Hence from (7), (8), and (9), the condition that the congruence l_1 be normal is

$$\frac{k\dfrac{\partial Q}{\partial u}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial P}{\partial u}, \quad \frac{-k\dfrac{\partial Q}{\partial v}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial P}{\partial v}, \tag{10}$$

where we must have

$$\frac{\partial^2 P}{\partial u \partial v} = \frac{\partial^2 P}{\partial v \partial u}. \tag{11}$$

It is readily found from (10), on making use of (8), that (11) is satisfied identically. Hence we have the following theorem.

THEOREM 1. *If an isotropic congruence l referred to S be inverted relative to C, the inverted congruence l_1 is normal.*

The condition that a congruence referred to S have the center of the sphere for its middle envelope is†

$$a = e^{-\lambda}\frac{\partial R}{\partial v}, \quad b = -e^{-\lambda}\frac{\partial R}{\partial u}, \tag{12}$$

where $R(u, v)$ is an arbitrary function. Let us again invert an isotropic congruence and ask when the inverted congruence l_1 will be of the type (12). From (7), (8), and (12) this condition will be

$$\frac{k\dfrac{\partial Q}{\partial u}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial R}{\partial v}, \quad \frac{k\dfrac{\partial Q}{\partial v}}{\left(\dfrac{\partial Q}{\partial u}\right)^2 + \left(\dfrac{\partial Q}{\partial v}\right)^2} = \frac{\partial R}{\partial u}, \tag{13}$$

* Foster, loc. cit., p. 173.

† Foster, loc. cit., p. 173.

where we must have $\partial^2 R/\partial u\partial v = \partial^2 R/\partial u\partial v$. As before we readily find on using (8) that this latter relation is satisfied identically. Hence we have the following theorem.

THEOREM 2. *If an isotropic congruence l referred to S be inverted relative to C, the inverted congruence l_1 is normal, and has for its middle envelope the center of S.*

The surfaces normal to l_1 have been considered by Appell.*

3. *The Rotation of Congruences.* We now refer the congruence l to the minimal surface S_1 whose lines of curvature are parametric. The equation whose roots t_1 and t_2 give the distances of the focal points from the tangent plane is†

$$q^2t^2 + q\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - ar_1 - br\right)t + \left(\frac{\partial a}{\partial v} - br_1\right)\left(\frac{\partial b}{\partial u} + ar\right) - \left(\frac{\partial a}{\partial u} + \xi - br\right)\left(\frac{\partial b}{\partial v} + \xi + ar_1\right) = 0.$$

Hence the necessary and sufficient condition that S_1 be the middle envelope of the congruence l is that $t_1+t_2=0$, or

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - ar_1 - br = 0. \tag{14}$$

On using (6) and multiplying by e^λ this relation becomes

$$\frac{\partial}{\partial u}(ae^\lambda) = \frac{\partial}{\partial v}(be^\lambda). \tag{15}$$

Let us set each member of (15) equal to $\partial^2 L/\partial u\partial v$, where L is an arbitrary function of u and v. Then the necessary and sufficient condition that S_1 be the middle envelope of l is

$$a = e^{-\lambda}\frac{\partial L}{\partial v}, \qquad b = e^{-\lambda}\frac{\partial L}{\partial u}. \tag{16}$$

On comparing this with the condition that a congruence referred to S_1 be normal,‡

* P. Appell, American Journal of Mathematics, vol. 10 (1888), p. 175.
† Foster, loc. cit., p. 170.
‡ Foster, loc. cit., p. 173.

$$a = e^{-\lambda}\frac{\partial P}{\partial u}, \quad b = -e^{-\lambda}\frac{\partial P}{\partial v}, \tag{17}$$

where $P(u, v)$ is an arbitrary function, we have the following theorem.

THEOREM 3. *If the lines of a normal congruence referred to S_1 be rotated through an angle $\pi/2$ about the corresponding normals, the middle envelope of the resulting congruence will be S_1.**

If a congruence is referred to the minimal surface S_2 with its asymptotic lines parametric, the equation for the distances from the tangent plane to the focal points is†

$$p^2t^2 - p\left(\frac{\partial a}{\partial v} + \frac{\partial b}{\partial u} + ar - br_1\right)t + \left(\frac{\partial a}{\partial v} - br_1\right)\left(\frac{\partial b}{\partial u} + ar\right) - \left(\frac{\partial a}{\partial u} + \xi - br\right)\left(\frac{\partial b}{\partial v} + \xi + ar_1\right) = 0.$$

The necessary and sufficient condition that S_2 be the middle envelope is

$$\frac{\partial a}{\partial v} + \frac{\partial b}{\partial u} + ar - br_1 = 0,$$

which becomes on using (6) and multiplying by e^{λ},

$$\frac{\partial}{\partial v}(ae^{\lambda}) = -\frac{\partial}{\partial u}(be^{\lambda}). \tag{18}$$

On setting each member of (18) equal to $\partial^2 M/\partial u\partial v$, where M is an arbitrary function of u and v, we find that the necessary and sufficient condition that S_2 be the middle envelope of the given congruence is

$$a = e^{-\lambda}\frac{\partial M}{\partial u}, \quad b = -e^{-\lambda}\frac{\partial M}{\partial v}. \tag{19}$$

The condition that a congruence referred to S_2 be normal is‡

* The direction of rotation is evidently immaterial, since P in (17) is arbitrary in sign.

† Foster, loc. cit., p. 172.

‡ Foster, loc. cit., p. 173.

$$a = e^{-\lambda}\frac{\partial P}{\partial v}, \quad b = e^{-\lambda}\frac{\partial P}{\partial u}, \tag{20}$$

where the function $P(u, v)$ is arbitrary; hence on comparing (19) and (20), we have the following theorem.

THEOREM 4. *If the lines of a normal congruence referred to S_2 be rotated through an angle $\pi/2$ about the corresponding normals, the middle envelope of the resulting congruence will be S_2.*

Let us now seek the conditions under which the normal congruence (17), referred to S_1, will remain normal after a rotation through an angle $\pi/2$. From (17) we must obviously have

$$\frac{\partial P}{\partial v} = \frac{\partial K}{\partial u}, \quad \frac{\partial P}{\partial u} = -\frac{\partial K}{\partial v},$$

where $K(u, v)$ is arbitrary. Hence from these equations the condition is that P satisfy Laplace's equation $\partial^2 P/\partial u^2 + \partial^2 P/\partial v^2 = 0$. We obtain the same result when we make a similar inquiry concerning the rotation of the normal congruence (20) referred to S_2. Consequently we have the theorem:

THEOREM 5. *If the lines of a normal congruence referred to S_1, (S_2), be rotated through an angle of $\pi/2$ about the corresponding normals, the resulting congruence will also be normal if the function P in* (17), [(19)], *is a solution of Laplace's equation, and will have S_1, (S_2), for its middle envelope.*

Since (16) is identical with (20), and (17) identical with (19), we have at once the following theorem.

THEOREM 6. *The point (a, b) defining a normal congruence referred to S_1, (S_2), when plotted with reference to the trihedral on S_2, (S_1), defines a congruence whose middle envelope is S_2, (S_1).*

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NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BECK (H.). Elementargeometrie. Teil 2. Leipzig, Akademische Verlagsgesellschaft, 1930. 10+184 pp.

BIEBERBACH (L.). Theorie der Differentialgleichungen. 3te neubearbeitete Auflage. (Die Grundlehren der Mathematischen Wissenschaften, Band 6.) Berlin, Springer, 1930. 14+399 pp.

BOLZANO (B.). Bernard Bolzano's Schriften. Herausgegeben von der Böhmischen Gesellschaft der Wissenschaften in Prag. Band 1: Funktionenlehre. Prag, 1930. 20+183+24+6 pp.

BOPP (K.). Drei Untersuchungen zur Geschichte der Mathematik. (Schriften der Strassburger wissenschaftlichen Gesellschaft in Heidelberg.) Berlin, de Gruyter, 1929. 66 pp.

BOREL (E.). Traité du calcul des probabilités et de ses applications. Tome I, fascicule II: Erreurs et moindres carrés, par R. Deltheil. Paris, Gauthier-Villars, 1930. 161 pp.

BRELOT (M.). See JULIA (G.).

BRESLICH (E. R.). The teaching of mathematics in secondary schools. Volume 1: Technique. Chicago, University of Chicago Press, 1930. 7+239 pp.

BRÜLL(E.). Erkenntniskritische Grundprobleme der Relativitätstheorie, Quanten- und Wellenmechanik. Breslau, Verlag Borgmeyer, 1929. 66 pp.

CARSLAW (H. S.). Plane trigonometry. 3d edition, revised. London, Macmillan, 1930. 180+330+12 pp.

CARTAN (E.). La théorie des groupes finis et continus et l'analysis situs. (Mémorial des Sciences Mathématiques, No. 41.) Paris, Gauthier-Villars, 1930.

COURANT (R.). Vorlesungen über Differential- und Integralrechnung. 2te, verbesserte Auflage. Band 1. Berlin, Springer, 1930. 410 pp.

CRATHORNE (A. R.) and LYTLE (E. B.). Trigonometry. New York, Holt, 1930. 10+95 pp.

CRAWLEY (E. S.) and EVANS (H. B.). Plane trigonometry. Revised edition. New York, F. S. Crofts, 1930. 6+178 pp.

CRELLE (E. L.). Rechentafeln welche alles Multiplizieren und Dividieren mit Zahlen unter Tausend ganz ersparen, bei grösseren Zahlen Rechnung erleichtern und sicherer machen. Neue Ausgabe besorgt von O. Seeliger. Mit Tafeln der Quadrate und Kubikzahlen von 1–1000. Berlin, de Gruyter, 1930. 7 pp. +1006 tables.

DANTZIG (T.). Number. The language of science. A critical survey written for the cultured non-mathematician. New York, Macmillan, 1930. 12+260 pp.

DELTHEIL (R.). See BOREL (E.).

DERUYTS (J.). Leçons d'algèbre. Déterminants; Théorie des équations. Bruxelles, 1929. 139 pp.

DICKSON (L. E.). Studies in the theory of numbers. Chicago, University of Chicago Press, 1930. 10+230 pp.

DRESDEN (A.). Solid analytic geometry and determinants. New York, Wiley, 1930. 10+310 pp.

DUBOURDIEU (J.). Sur les réseaux de courbes et de surfaces. (Hamburger Mathematische Einzelschriften, No. 9.) Leipzig, Teubner, 1930. 96 pp.

DURELL (C. V.) and ROBSON (A.). Advanced trigonometry. London, Bell, 1930. 8+335 pp.

EVANS (H. B.). See CRAWLEY (E. S.).

FICK (E.). Aufgabensammlung über Differentialgleichungen. München, Oldenbourg, 1930. 7+177 pp.

FULCO (P.). Quozienti e resti nella divisione di frazioni. Congruenze a modulo frazionario. Memoria IV. Civitavecchia, 1929. 104 pp.

GARNIER (E.). La matematica che serve: algebra, trigonometria, geometria analitica. Milano, Hoepli, 1930. 12+522 pp.

HESSENBERG (G.). Grundlagen der Geometrie. Herausgegeben von W. Schwan. (Göschens Lehrbücherei.) Berlin, de Gruyter, 1930. 143 pp.

HILBERT (D.). Grundlagen der Geometrie. 7te umgearbeitete und vermehrte Auflage. (Wissenschaft und Hypothese, VII.) Leipzig, Teubner, 1930. 8+326 pp.

HOHEISEL (G.). Gewöhnliche Differentialgleichungen. 2te, verbesserte Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1930. 159 pp.

IBBETSON (W. S.). Preliminary mathematics for engineers. London, Pitman, 1930. 11+152 pp.

JACKSON (D.). The theory of approximation. (American Mathematical Society Colloquium Publications, vol. 11.) New York, American Mathematical Society, 1930. 8+178 pp.

JULIA (G.). Principes géométriques d'analyse. Partie 1. Leçons faites à la Sorbonne récueillies et rédigées par M. Brelot et R. de Possel. (Cahiers Scientifiques, No. 6.) Paris, Gauthier-Villars, 1930. 6+116 pp.

LANDAU (E.). Grundlagen der Analysis. (Das Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen.) Leipzig, Akademische Verlagsgesellschaft, 1930. 134 pp.

LENNES (N. J.) and MERRILL (A. S.). Plane analytic geometry. New York, Harper, 1929. 15+305 pp.

LINDELÖF (E. L.). Commentationes in honorem Ernesti Leonardi Lindelöf a discipulis editae. Helsinki, Suomalainen Tiedeakatemia, 1929. 6+360 pp.

LYTLE (E. B.). See CRATHORNE (A. R.).

MERRILL (A. S.). See LENNES (N. J.).

METZLER (W. H.). See MUIR (T.).

MIKESH (J. S.). See MILES (E. J.).

MILES (E. J.) and MIKESH (J. S.). Calculus. New York, McGraw-Hill, 1930. 14+638 pp.

MOHRMANN (H.). Einführung in die nicht-euklidische Geometrie. Leipzig, Akademische Verlagsgesellschaft, 1930. 12+126 pp.

MOULTON (F. R.). Differential equations. New York, Macmillan, 1930. 16+395 pp.

MUIR (T.). Treatise on the theory of determinants. Revised and greatly

enlarged by W. H. Metzler. Menasha, Wis., Banta Publishing Company, 1930. 774 pp.

NOACK (A.). Über eine zum projektiven, einseitigen R_n homöomorphe Mannigfaltigkeit, die ein- und zweiseitig erscheint. (Dissertation, Kiel.) Göttingen, 1929. 35 pp.

D'OCAGNE (M.). Cours de géométrie. Paris, Gauthier-Villars, 1930. 6+429 pp.

PAPELIER (G.). Eléments de trigonométrie sphérique. Paris, Vuibert, 1930. 165 pp.

PASCH (M.). Der Ursprung des Zahlbegriffs. Neudruck. Berlin, Springer, 1930. 50 pp.

PHILLIPS (E. G.). A course of analysis. Cambridge, University Press, 1930. 8+361 pp.

DE POSSEL (R.). See JULIA (G.).

POTTHOF (R.). Die anschauliche Natur der geometrischen Grundbegriffe. Münster, Aschendorff'sche Verlagsbuchhandlung, 1930. 28 pp.

REY (A.). La science orientale avant les Grecs. Paris, La Renaissance du Livre, 1930. 495 pp.

ROBSON (A.). See DURELL (C. V.).

ROGOSINSKI (W.). Fouriersche Reihen. (Sammlung Göschen.) Berlin, de Gruyter, 1930. 135 pp.

SAUER (R.). Über nichtbeschränkte P-Formen. Auszug aus einer Inaugural-Dissertation, Kiel. Kiel, C. Schaidt, 1929. 4 pp.

SCHEFFERS (G.). Allerhand aus der zeichnenden Geometrie. Leipzig, Quelle und Meyer, 1930. 62 pp.

SCHÖBE (W.). Beiträge zur Funktionentheorie in nichtarchimedisch bewerteten Körpern. (Band 42 des Universitäts Archivs.) Münster, Helios-Verlag, 1930. 61 pp.

SCHOLZ (A.). Reduktion der Konstruktion von Körpern mir zweistufiger (metabelscher) Gruppe. Berlin, de Gruyter, 1929. 15 pp.

SCHWAN (W.). See HESSENBERG (G.).

SEELIGER (O.). See CRELLE (A. L.).

SEGRE (B.). See SEVERI (F.).

SEVERI (F.). Conferenze di geometria algebrica, raccolte da B. Segre. Roma, Stab. Tipo-Lit. del Genia Civile, 1927–29. 4+392 pp.

THORNDYKE (L.). Science and thought in the fifteenth century. New York, Columbia University Press, 1929. 14+387 pp.

TRINKNER (O.). Über das asymptotische Verhalten der Entwicklungskoeffizienten gewisser Funktionen der Form $e^{f(x)}$. (Dissertation, Tübingen.) Leipzig-Borna, R. Noske, 1930.

WIELEITNER (H.). Algebraische Kurven. Durchgesehener Neudruck. Teil 1: Gestaltliche Verhältnisse. (Sammlung Göschen.) Berlin, de Gruyter, 1930. 146 pp.

WIMAN (A.). Opuscula mathematica Andreae Wiman dedicata. Festskrift Anders Wiman. Upsala, Lunquistska Bokhandeln, 1930.

WOLF (A.). Textbook of logic. London, Allen and Unwin, 1930. 407 pp.

YOUNG (J. W.). Projective geometry. (Carus Mathematical Monographs, No. 4.) Chicago, Open Court, 1930. 10+185 pp.

PART II. APPLIED MATHEMATICS

ABRAHAM (M.). Theorie der Elektrizität. Vollständig neubearbeitet von R. Becker. 8te Auflage. Band 1: Einführung in die Maxwellsche Theorie der Elektrizität. Leipzig, Teubner, 1930. 8+226 pp.

ADAM (N. K.). The physics and chemistry of surfaces. Oxford, Clarendon Press, 1930. 10+332 pp.

ARISTOTLE. The works of Aristotle. Translated into English. Physica. By R. P. Hardie and R. K. Gaye. Oxford, Clarendon Press, 1930. 8+260 pp.

AUSTIN (E.). See SHAW (N.).

BARTLETT (A. C.). The theory of electrical artificial lines and filters. London, Chapman and Hall, 1930. 9+155 pp.

BECKER (R.). See ABRAHAM (M.).

BRESSON (C.). Appareillage électrique haute tension, théorie, construction et applications. Paris, Dunod, 1929. 4+460 pp.

BUCKINGHAM (J.). Matter and radiation with particular reference to the detection and uses of the infra-red rays. London, Oxford University Press, 1930. 12+144 pp.

BUSH (V.). See TIMBIE (W. H.).

CAMICHEL (C.). Leçons sur les conduites. Paris, Gauthier-Villars, 1930. 6+101 pp.

CASTELFRANCHI (G.). Physique moderne. Paris, A. Blanchard, 1930. 2+660 pp.

CHAZY (J.). La théorie de la rélativité et la mécanique céleste. Tome 1 et tome 2. Paris, Gauthier-Villars, 1930. 8+261+258 pp.

CROWTHER (J. A.). A manual of physics. 3d edition. London, Oxford University Press, 1930. 562 pp.

DAVIES (C. W.). The conductivity of solutions and the modern dissociation theory. London, Chapman and Hall, 1930. 8+204 pp.

DIGWEED (E. N.). Hydraulics. London, Association of Engineering and Shipbuilding Draughtsmen, 1930. 48 pp.

EBERHARD (G.), KOHLSCHÜTTER (A.), und LUDENDORFF (H.), herausgegeben von. Handbuch der Astrophysik. Band 2: Grundlagen der Astrophysik, Teil 2, I; Band 3: Grundlagen der Astrophysik, Teil 3, I. Berlin, Springer, 1930. 11+430+10+473 pp.

ERRETH (A.). Die Aberration und die Geschwindigkeit des Lichtstrahles. Übersetzung aus dem Ungarischen, Budapest, Druck der Athenaeum, 1929. 2+49 pp.

ESTERMANN (I.). Elektrischen Dipolmomente von Molekülen. (Ergebnisse der exakten Naturwissenschaften.) Berlin, Springer, 1929.

FISHER (R. A.). Statistical methods for research workers. 3d edition, revised and enlarged. London and Edinburgh, Oliver and Boyd, 1930. 14+283 pp.

FÖPPL (L.). Aufgaben aus technischer Mechanik. Statik, Festigkeitslehre, Dynamik. München und Berlin, Oldenbourg, 1930. 7+188 pp.

GAYE (R. K.). See ARISTOTLE.

GOUARD (E.) et HIÉRNAUX (G.). Cours élémentaire de mécanique industrielle. 3e édition. Tome 2. Paris, Dunod, 1930. 4+367 pp.

GOUDSMIT (S.). See PAULING (L.).

GRIFFITH (R. O.) and MCKEOWN (A.). Photo-processes in gaseous and liquid systems. London and New York, Longmans, 1929. 8+691 pp.

GRÜBLER (M.). Was ist Gewicht? Dresden, Steinkopff, 1930. 32 pp.

HAAS (A.). Einführung in die theoretische Physik. 5te und 6te, abermals völlig umgearbeitete und wesentlich vermehrte Auflage. Band 1. Berlin, de Gruyter, 1930. 10+396 pp.

HABANN (E.). Die neuere Entwicklung der Hochfrequenztelephonie und -telegraphie auf Leitungen. Braunschweig, Vieweg, 1929. 8+167 pp.

HAISSINSKY (M.). L'atomistica moderna e la chimica. Milano, Hoepli, 1930. 14+315 pp.

HALPERN (O.) und THIRRING (H.). Die Grundgedanken der neueren Quantentheorie. Berlin, Springer, 1929. 241 pp.

HARDIE (R. P.). See ARISTOTLE.

HARMS (F.). See WIEN (W.).

HEINZELMANN (H.). Die elektrischen Kabel. (Sammlung Göschen.) Berlin, de Gruyter, 1930. 133 pp.

HEISENBERG (W.). Die physikalischen Prinzipien der Quantentheorie. Leipzig, Hirzel, 1930. 8+120 pp.

HENSELING (R.). Der neuentdeckte Himmel. Das astronomische Weltbild gemäss jüngster Forschung. Berlin, Atlantisverlag, 1930. 124 pp.

HEYL (P. R.). New frontiers of physics, New York, Appleton, 1930. 9+170 pp.

HIÉRNAUX (E.). See GOUARD (E.).

HOAG (J. B.). Electron physics. London, Chapman and Hall, 1930. 9+208 pp.

HOFFMAN (G.). See WIEN (W.).

HOLZINGER (K. J.). Statistical résumé of the Spearman two-factor theory. Chicago, University of Chicago Press, 1930. 43 pp.

JÄGER (G.). Theoretische Physik. 6te Auflage, 1ter, 2ter, 3ter, 4ter Band, und 4te Auflage, 5ter Band. (Sammlung Göschen.) Berlin, de Gruyter, 1930.

JAMESON (A. H.) and ORMSBY (M. T. M.). Mathematical geography. Volume 2: Simple astronomical and trigonometric surveying, and the more advanced study of map projections. London, Pitman, 1929. 8+160 pp.

JELLINEK (K.). Lehrbuch der physikalischen Chemie. Band 3, Die Lehre von der Statik homogener und heterogener Gasreaktionen, Lieferung 8. Stuttgart, Enke, 1930.

KALTENBACH (P.) und MELDAU (H.). Physik für Seefahrer. Braunschweig, Vieweg, 1930. 8+408 pp.

KAUFMANN (W.). Statik der Tragwerke. 2te verbesserte und ergänzte Auflage. Berlin, Springer, 1930. 8+322 pp.

KIRCHBERGER (P.). La théorie atomique. Son histoire et son développement. Traduit par Marcel Thiers. Paris, Payot, 1930.

KIRCHNER (F.). See WIEN (W.).

KNAB (H. J.). Übersicht über Kinematik. Getriebelehre mit besonderer Berücksichtigung der Getriebedynamik sowie der Schwingungsgetriebe und Getriebeschwingungen. 2te erweiterte und völlig umgearbeitete Auflage. Nürnberg, Selbstverlag, 1930. 246 pp.

KOHLSCHÜTTER (A.). See EBERHARD (G.).

KREMANN (R.) und MÜLLER (R.). Elektromotorische Kräfte. Elektrolyse und Polarisation. Teil 1: Elektromotorische Kräfte. (Handbuch der allgemeine Chemie, Band 8.) Leipzig, Akademische Verlagsgesellschaft, 1930. 16+891 pp.

LODGE (O.). Beyond physics, or the idealisation of mechanism. Being a survey and attempted extension of modern physics in a philosophical and psychical direction. London, Allen and Unwin, 1930. 172 pp.

——— Ether and reality. Many functions of the ether of space. London, Hodder and Stoughton, 1930. 180 pp.

LONGINESCU (I. N.). Essai sur les principes de la thermodynamique. Paris, Société d'Editions Sciéntifiques, 1929. 82 pp.

LUDENDORFF (H.). See EBERHARD (G.).

LUPTON (A. W.). Aids to the mathematics of pharmacy. London, Bailliere, Tindall, and Cox, 1930. 96 pp.

MCKEOWN (A.). See GRIFFITH (R. O.).

MACMILLAN (W. D.). The theory of the potential. New York, McGraw-Hill, 1930. 14+469 pp.

DE MARCHI (G.). Idraulica. Volume 1. Milano, Hoepli, 1930.

MATOSSI (F.). See SCHAEFER (C.).

MAZET (R.). See PAINLEVÉ (P.).

MELDAU (H.). See KALTENBACH (P.).

METRAL (A.). See PAINLEVÉ (P.).

MÜLLER (R.). See KREMANN (R.).

NICOL (G.). Ship construction and calculation. Glasgow, Brown, Son and Ferguson, 1930. 580 pp.

ORMSBY (M. T. M.). See JAMESON (A. H.).

PAINLEVÉ (P.). Leçons sur la résistance des fluides non visqueux. Rédigées par A. Metral et R. Mazet. Partie 1, rédigée par A. Metral. Paris, Gautheir-Villars, 1930. 7+183 pp.

PAINLEVÉ (P.) et PLATRIER (C.). Cours de mécanique. Mécanique des solides indéformables. Mécanique des milieux continus déformables. Théorie sommaire des machines et de l'aviation. Les mécaniques de Newton et d'Einstein. Paris, Gauthier-Villars, 1929. 8+644 pp.

PARKER (R. L.). Kristallzeichnen. Berlin, Gebrüder Borntraeger, 1929. 6+112 pp.

PAULING (L.) and GOUDSMIT (S.). The structure of line spectra. New York, McGraw-Hill, 1930. 262 pp.

PETER (H.). Grenzen der Statistik und Konjunkturforschung. Ein Beitrag zur Kritik der Wirtschaftsprognose. Bonn, K. Schroeder, 1930. 73 pp.

PINK (E. N.). The electric and magnetic circuits. Alternating current and direct current. London, Pitman, 1930. 7+124 pp.

PLANCK (M.). Einführung in die theoretische Physik. Band 5: Einführung in die Theorie der Wärme. Leipzig, Hirzel, 1930. 7+251 pp.

——— Vorlesungen über Thermodynamik. 9te Auflage. Berlin, de Gruyter, 1930. 9+288 pp.

PLATRIER (C.). See PAINLEVÉ (P.).

RABINOVITSCH (E.) und THILO (E.). Periodisches System. Geschichte und Theorie. Stuttgart, Enke, 1930. 12+302 pp.

RIDEAL (E. K.). An introduction to surface chemistry. 2d edition, revised and enlarged. Cambridge, University Press, 1930.

RIGGS (N. C.). Applied mechanics. New York, Macmillan, 1930. 10+328 pp.

ROBSON (A.) and TRIMBLE (C. J. A.). Examples in mechanics. London, Bell, 1930. 7+158 pp.

ROSENTHAL (J.). Das Jahrhundert der Strahlen. Leipzig, Georg Thieme, 1930. 29 pp.

RUSSELL (A. H.). Checking calculations. For accountants, clerks and teachers. London and New York, Gregg, 1930. 64 pp.

SACK (H.). Dipolmoment und Molekularstruktur. (Ergebnisse der exakten Naturwissenschaften.) Berlin, Springer, 1929.

SCHAEFER (C.) und MATOSSI (F.). Das ultrarote Spektrum. (Struktur der Materie in Einzeldarstellungen.) Berlin, Springer, 1930. 6+400 pp.

SCHILLER (L.). See WIEN (W.).

SCHUMANN (W. O.). See WIEN (W.).

SEN (D. K.). Hydrostatics. London, Macmillan, 1930. 6+242 pp.

SHAW (N.), assisted by AUSTIN (E.). Manual of meteorology. Volume 3: The physical processes of the weather. Cambridge, University Press, 1930. 28+445 pp.

SMITH (C. J.). See WULF (T.).

SULLIVAN (J. W. N.). Present-day astronomy. London, George Newnes, 1930. 144 pp.

SUTCLIFFE (G. E.). The new astronomy and cosmic physiology. An introduction to the subject. London, Rider, 1930. 139 pp.

THIBAUD (J.). Les rayons X. Théorie et applications. Paris, Armand Colin, 1930. 216 pp.

THIERS (M.). See KIRCHBERGER (P.).

THILO (E.). See RABINOVITSCH (E.).

THIRRING (H.). See HALPERN (O.).

TIMBIE (W. H.) and BUSH (V.). Principles of electrical engineering. 2d edition. London, Chapman and Hall, 1930. 7+595 pp.

TRIMBLE (C. J. A.). See ROBSON (A.).

WEBB (G. W.). See WILSON (P.).

WIEN (W.) und HARMS (K.), heraugsgegeben von. Handbuch der Experimentalphysik. Band 4, Hydro- und Aerodynamik, Teil 3: Technische Anwendungen, herausgegeben von L. Schiller. Leipzig, Akademische Verlagsgesellschaft, 1930. 10+557 pp.

——— Handbuch der Experimentalphysik. Band 10: Elektrostatik, von G. Hoffmann; Hochspannungstechnik, von W. O. Schumann. Leipzig, Akademische Verlagsgesellschaft, 1930. 10+576 pp.

——— Handbuch der Experimentalphysik. Band 24, Teil 1: Allgemeine Physik der Röntgenstrahlen, von F. Kirchner. Leipzig, Akademische Verlagsgesellschaft, 1930. 10+548 pp.

WILLIAMS (J. W.). Molekulare Dipolmomente und ihre Bedeutung für die chemische Forschung. Berlin, Gebrüder Borntraeger, 1930. 3+66 pp.

WILSON (P.) and WEBB (G. W.). Modern gramophones and electrical reproducers. London, Cassell, 1929. 16+272 pp.

WOLKOWITSCH (D.). Applications de la géométire à la stabilité des constructions. Tome 2. Paris, Doin, 1930. 270 pp.

WULF, (T.). Modern physics. A general survey of its principles. Translated from the second German edition by C. J. Smith. London, Methuen, 1930. 11+469 pp.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 25, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than October 4. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Secretary not later than September 7. By invitation of the program committee, Professor M. H. Stone will deliver an address on group representation in Hilbert space.

COLUMBIA, MISSOURI, November 28-29, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than November 6. All abstracts received by that date will appear in the Bulletin in advance of the meeting. On Saturday morning, Professor C. N. Moore will deliver an address by invitation of the program committee, on types of series and types of summability; and Professor R. E. Langer will deliver an address on the zeros of exponential sums and of certain related functions.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, November 29, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, Berkeley, California, not later than November 1. All abstracts received by that date will appear in the Bulletin in advance of the meeting.

CLEVELAND, OHIO, ANNUAL MEETING, December 29, 1930 to January 1, 1931.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than December 1. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than November 7.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-458 Ahnaip St., Menasha, Wis., or 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, G. W. MULLINS, 501 West 116th Street, New York.

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